Extended Solutions for Instructors

for the Book

An Introduction to Partial Differential Equations

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Chapter 1

1.1 (a) Write $u_x = af'$, $u_y = bf'$. Therefore, a and b can be any constants such that $a + 3b = 0$.

1.3 (a) Integrate the first equation with respect to x to get $u(x,y) = x^3y + xy + F(y)$, where $F(y)$ is still undetermined. Differentiate this solution with respect to y and compare to the equation for u_y to conclude that F is a constant function. Finally, using the initial condition $u(0,0) = 0$, obtain $F(y) = 0$.

(b) The compatibility condition $u_{xy} = u_{yx}$ does not hold. Therefore, there does not exist a function u satisfying both equations.

1.5 Differentiate $u = f(x + p(u)t)$ by t:

$$u_t = f'(x + p(u)t)\,(p(u) + tp'(u)u_t) \Rightarrow (1 - tf'p')u_t = pf'.$$

The expression $1 - tf'p'$ cannot vanish on a t-interval, otherwise, $pf' = 0$ there. But this is a contradiction, since if either p or f' vanishes in this interval, then $tf'p' = 0$ there. Therefore, we can write

$$u_t = \frac{pf'}{1 - tp'f'}.$$

Similarly,

$$u_x = \frac{f'}{1 - tp'f'},$$

and the claim follows.

(a) Substituting $p = k$ (for a constant k) into $u = f(x + p(u)t)$ provides the explicit solution $u(x,t) = f(x + kt)$, where f is any differentiable function.

(b), (c) Equations (b) and (c) do not have such explicit solutions. Nevertheless, if we select $f(s) = s$, we obtain that (b) is solved by $u = x + ut$ that can be written explicitly as $u(x,t) = x/(1-t)$, which is well-defined if $t \neq 1$.

1.7 (a) Substitute $v(s,t) = u(x,y)$, and use the chain rule to get

$$u_x = v_s + v_t, \qquad u_y = -v_t,$$

and

$$u_{xx} = v_{ss} + v_{tt} + 2v_{st}, \qquad u_{xy} = -v_{tt} - v_{st}, \qquad u_{yy} = v_{tt}.$$

Therefore, $u_{xx} + 2u_{xy} + u_{yy} = v_{ss}$, and the equation becomes $v_{ss} = 0$.

(b) The general solution is $v = f(t) + sg(t)$, where f and g are arbitrary differentiable functions. Thus, $u(x,y) = f(x-y) + xg(x-y)$ is the desired general solution in the (x,y) coordinates.

(c) Proceeding similarly, we obtain for $v(s,t) = u(x,y)$:

$$\begin{aligned} u_x &= v_s + 2v_t, \qquad\qquad u_y = v_s, \\ u_{xx} &= v_{ss} + 4v_{tt} + 4v_{st}, \quad u_{yy} = v_{tt}, \qquad u_{xy} = v_{ss} + 2v_{st}. \end{aligned}$$

Hence, $u_{xx} - 2u_{xy} + 5u_{yy} = 4(v_{ss} + v_{tt})$, and the equation is $v_{ss} + v_{tt} = 0$.

Chapter 2

2.1 (a), (b) The characteristic equations are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 1, \quad \frac{\mathrm{d}u}{\mathrm{d}t} = 0.$$

Therefore, the characteristics are $y = x + c$, and the solution is $u(x, y) = f(x - y) + y$.

2.3 (a) The characteristic equations are

$$x_t = x, \quad y_t = y, \quad u_t = pu.$$

The solution is

$$x(t, s) = x_0 \mathrm{e}^t, \qquad y(t, s) = y_0 \mathrm{e}^t, \qquad u(t, s) = u_0 \mathrm{e}^{pt}.$$

Thus, the projections on the (x, y) plane are the curves $x/y =$ constant.
(b) The solution is $u(x, y) = (x^2 + y^2)^2$. It is a unique solution since the transversality condition holds.
(c) The initial curve $(s, 0, s^2)$ is a characteristic curve (see the characteristic equations). Thus, there exist infinitely many solutions of the form $u(x, y) = x^2 + ky^2$, where $k \in \mathbb{R}$.

2.5 (a) The projection on the (x, y) plane of each characteristic curve has a positive direction and it propagates with a strictly positive speed in the square. Therefore, it intersects the boundary of D at exactly 2 points.
(b) Suppose that u is positive on ∂D, and suppose that $u \le 0$ at some point in D. Consider the characteristic line through this point. Since u on each characteristic line equals $u(t) = f(s)\mathrm{e}^{-t}$, it follows that $u \le 0$ at the two points where the projection of this line intersects the boundary of the square, but this contradicts our assumption.
(c) Let (x_0, y_0) be the point in D where u attains a minimum. Since $\nabla u(x_0, y_0) = 0$, it follows from the PDE that $u(x_0, y_0) = 0$.
(d) If $u(x, y) \ngeq m$ for all $(x, y) \in D$, then u attains its global minimum in $\bar{D}$ at some $(x_0, y_0) \in D$, and by part (c), $u(x_0, y_0) = 0$. But this contradicts part (b).

2.7 Solving the characteristic equations together with the initial condition we find

$$(x(t, s), y(t, s), u(t, s)) = (t + s, t, 1/(1 - t)).$$

Therefore $u = 1/(1-y)$. Alternatively, since the the initial condition does not depend at all on x, one can guess that the solution does not depend on x either. The problem is then reduced to the ODE $\mathrm{d}u/\mathrm{d}y = u^2, \;\; u(0) = 1$, whose solution is indeed $1/(1-y)$. Since the transversality condition holds, the uniqueness is guaranteed.

2.9 (a) The vector tangent to the initial curve is $(1, 0, \cos s)$.

The characteristic equations are

$$x_t = u, \qquad y_t = 1, \qquad u_t = -\frac{1}{2}u.$$

The *direction* of the characteristic curves on the initial curve is $(\sin s, 1, -\frac{1}{2}\sin s)$. Since the projection of these directions on the (x, y) plane are not parallel for all $-\infty < s < \infty$, we conclude that the transversality condition holds, and there exists a unique solution near the initial curve.
(b) Solving the characteristic equations we obtain

$$x(t,s) = s - 2\sin s\left(\mathrm{e}^{-t/2} - 1\right), \quad y(t,s) = t, \quad u(t,s) = \sin s\,\mathrm{e}^{-t/2}.$$

(c) To find the solution passing through Γ_1, we solve the characteristic equations together with the initial curve $(s, s, 0)$. We obtain:

$$x(t,s) = s, \qquad y(t,s) = s + t, \qquad u(t,s) = 0,$$

namely, $u(x, y) = 0$.
(d) Notice that the required curve must be a characteristic curve. Since it passes through the origin $x = y = u = 0$, we obtain from the characteristic equations

$$x = 0, \quad y = t, \quad u = 0.$$

Thus, the curve is exactly the y axis.

2.11 The characteristic equations and the initial conditions are given by

$$x_t = y^2 + u, \quad y_t = y, \quad u_t = 0, \tag{12.4}$$

and

$$x(0,s) = \frac{s^2}{2}, \quad y(0,s) = s, \quad u(0,s) = 0, \tag{12.5}$$

respectively. Computing the Jacobian we find that $J \equiv 0$. It is easy to check that $u \equiv 0$ solves the problem. Therefore, there exist infinitely many solutions. We compute for instance another solution. For this purpose we define a new Cauchy problem

$$(y^2 + u)u_x + yu_y = 0, \quad u(x,1) = x - \frac{1}{2}.$$

Now the Jacobian satisfies $J \equiv 1$. The parametric form of the solution is

$$\begin{aligned} x(t,s) &= (s - \frac{1}{2})t + \frac{1}{2}\mathrm{e}^{2t} + s - \frac{1}{2}, \\ y(t,s) &= \mathrm{e}^t, \\ u(t,s) &= s - \frac{1}{2}. \end{aligned}$$

It is convenient in this case to express the solution as a graph of the form

$$x(y,u) = \frac{1}{2}y^2 + u\ln y + u.$$

2.13 The characteristic equations are

$$x_t = u, \qquad y_t = x, \qquad u_t = 1. \tag{12.6}$$

First, verify that the transversality condition is violated at every point, and that the problem has infinitely many solutions. We obtain one such solution through an "intelligent guess". We seek a solution of the form $u = u(x)$. Substituting $u(x)$ into the equation and the initial data we obtain $u(x) = \sqrt{2(x-1)}$. To find another solution we define a new Cauchy problem, such that the new initial curve identifies with the original initial curve at the point $s = 1$.

$$uu_x + xu_y = 1, \quad u(x + \frac{3}{2}, \frac{7}{6}) = 1.$$

The parametric representation of the solution to the new problem is given by

$$\begin{aligned} x(t,s) &= \frac{1}{2}t^2 + t + d + \frac{3}{2}, \\ y(t,s) &= \frac{1}{6}t^3 + \frac{1}{2}t^2 + (t + d + \frac{3}{2})t + \frac{7}{6}, \\ u(t,s) &= t + 1. \end{aligned}$$

We can eliminate now

$$\begin{aligned} t &= u - 1, \\ d &= x - \frac{3}{2} - \frac{(u-1)^2}{2} - (u-1). \end{aligned}$$

Thus, the solution to the original problem is given by

$$y(x,u) = \frac{1}{6}(u-1)^3 + \frac{1}{2}(u-1)^2 + (u-1)\left[x - \frac{1}{2}(u-1)^2 - (u-1)\right] + \frac{7}{6}.$$

2.15 (a) We write the characteristic equations:

$$x_t = x + y^2, \qquad y_t = y, \qquad u_t = 1 - \left(\frac{x}{y} - y\right)u,$$

where the initial conditions are given by

$$x(0,s) = s, \qquad y(0,s) = 1, \qquad u(0,s) = 0.$$

Notice that the first two equations can be solved independently of the third equation. We find
$$y(t,s) = \mathrm{e}^t, \qquad x(t,s) = s\mathrm{e}^t + \mathrm{e}^t(\mathrm{e}^t - 1),$$
and invert these relations to get
$$t = \ln y, \qquad s = \frac{x}{y} - y + 1.$$

Substituting this result into the third equation gives
$$u_t = -(s-1)u + 1,$$
implying
$$u(t,s) = \frac{1 - \mathrm{e}^{(s-1)t}}{s-1},$$
and then
$$u(x,y) = y\frac{1 - y^{x/y-y}}{x - y^2}.$$
(b) and (d). The transversality condition is equivalent here to $(s+1)\times 0 - 1 = -1 \neq 0$. Therefore, this condition holds for all s. The explicit solution shows that u is not defined at the origin. This does not contradict the local existence theorem, since this theorem only guarantees a solution in a neighborhood of the original curve ($y = 1$).

2.17 (a) The characteristic equations are
$$x_t = x, \quad y_t = 1, \quad u_t = 1.$$
The solution is
$$x(t,s) = x_0\mathrm{e}^t, \quad y(t,s) = y_0 + t, \quad u(t,s) = u_0 + t.$$
The characteristic curve passing through the point $(1,1,1)$ is $(\mathrm{e}^t, 1+t, 1+t)$.
(b) The direction of the projection of the initial curve on the (x,y) plane is $(1,0)$. The direction of the projection of the characteristic curve is $(s,1)$. Since the directions are not parallel, there exists a unique solution. To find this solution, we substitute the initial curve into the formula for the characteristic curves, and find
$$x(t,s) = s\mathrm{e}^t, \quad y(t,s) = t, \quad u(t,s) = \sin s + t.$$
Eliminating s and t we get $s = x/\mathrm{e}^y$. The explicit solution is $u(x,y) = \sin(x/\mathrm{e}^y) + y$. It is defined for all x and y.

2.19 The characteristic equations and their solutions are
$$x_t = x^2, \qquad y_t = y^2, \qquad u_t = u^2,$$
$$x(t,s) = \frac{x_0}{1 - x_0 t}, \qquad y(t,s) = \frac{y_0}{1 - y_0 t}, \qquad u(t,s) = \frac{u_0}{1 - u_0 t}.$$

The projection of the initial curve on the (x, y) plane is in the direction $(1, 2)$. The direction of the projection of the characteristic curve (for points on the initial curve) is $s^2(1, 4)$. The directions are not parallel, except at the origin where the characteristic direction is degenerate.

Solving the Cauchy problem gives:

$$x(t, s) = \frac{s}{1 - st}, \quad y(t, s) = \frac{2s}{1 - 2st}, \quad u(t, s) = \frac{s^2}{1 - s^2 t}.$$

Eliminating s and t we find

$$u(x, y) = \frac{x^2 y^2}{4(y - x)^2 - xy(y - 2x)}.$$

Notice that the solution is not defined on the curve $4(x-y)^2 = xy(y-2x)$ that passes through the origin.

2.21 The characteristic equations are

$$x_t = x, \qquad y_t = -y, \qquad u_t = u + xy.$$

The curve $(1, 1, 2s)$ is tangent to the initial data. On the other hand, the characteristic direction along the initial curve is $(s, -s, 2s^2)$. Clearly the projections of these direction vectors on the (x, y) plane are not parallel for $1 \leq s \leq 2$, and thus the transversality condition holds.
To construct a solution we substitute the initial curve into the characteristic equations, and find that

$$x(t, s) = s\mathrm{e}^t, \quad y(t, s) = s\mathrm{e}^{-t}, \quad u(t, s) = 2s^2\mathrm{e}^t - s^2.$$

Eliminating $s^2 = xy$, $\mathrm{e}^t = \sqrt{x/y}$, we get

$$u(x, y) = 2x^{3/2}y^{1/2} - xy.$$

This solution is defined only for $y > 0$.

2.23 The characteristic equations and the initial conditions are

$$t_\tau = 1, \quad x_\tau = c, \quad u_\tau = -u^2,$$

and

$$t(0, s) = 0, \quad x(0, s) = s, \quad u(0, s) = s.$$

Let us check the transversality condition:

$$J = \begin{vmatrix} 1 & c \\ 0 & 1 \end{vmatrix} = 1 \neq 0. \tag{12.7}$$

We solve the equations and get

$$t(\tau, s) = \tau, \qquad x(\tau, s) = c\tau + s, \qquad u(\tau, s) = \frac{s}{1 + \tau s}.$$

Therefore, the solution is

$$u(x,t) = \frac{x - ct}{1 + t(x - ct)}.$$

The observer that starts at the point x_0 sees the solution

$$u(x_0 + ct, t) = \frac{x_0}{1 + x_0 t}.$$

Therefore, if $x_0 > 0$, the observed solution decays, while if $x_0 < 0$ the solution explodes in a finite time. We finally remark that if $x_0 = 0$, then the solution is 0.

2.25 The transversality condition is violated identically. However the characteristic direction is $(1, 1, 1)$, and so is the direction of the initial curve. Therefore the initial curve is itself a characteristic curve, and there exist infinitely many solutions. To find solutions, consider the problem

$$u_x + u_y = 1, \quad u(x, 0) = f(x),$$

for an arbitrary f satisfying $f(0) = 0$. The solution is $u(x, y) = y + f(x - y)$. It remains to fix five choices for f.

2.27 (a) Use the method of Example 2.13. Since $(a, b, c) = (u, 1, 1)$, identify $\vec{P}_1 = (-1, 0, u)$ and $\vec{P}_2 = (0, 1, -1)$. Therefore, $\psi(x, y, u) = -x + u^2/2$, and $\phi(x, y, u) = u - y$, and the general solution is $f(u - y) = u^2/2 - x$ for an undetermined function f. The initial condition then implies

$$u(x, y) = \frac{6y - y^2 - 2x}{2(3 - y)}.$$

(b) A straightforward calculation verifies $u(3x, 2) = 4 - 3x$.
(c) The transversality condition holds in this case. Therefore the problem has a unique solution. From (b) we obtain that the solution is the same as in (a).

Chapter 3

3.1 (a) We know that the equation is parabolic. Therefore, it is easy to see that the required transformation satisfies

$$y = t, \quad x = \frac{s-t}{3}.$$

(b) Integrating twice with respect to t, we get

$$v(s,t) = \frac{1}{324}st^4 - \frac{1}{540}t^5 + t\phi(s) + \psi(s),$$

where ψ, ϕ are integration factors. Returning to the original variables, we obtain

$$u(x,y) = \frac{1}{324}(3x+y)y^4 - \frac{1}{540}y^5 + y\phi(3x+y) + \psi(3x+y).$$

(c) Using the initial conditions we infer that

$$\begin{aligned} u(x,0) &= \psi(3x) = \sin x \Rightarrow \psi(x) = \sin(x/3), \\ u_y(x,0) &= \phi(3x) + \psi'(3x) = \cos x \Rightarrow \phi(x) = \cos(x/3) - \frac{1}{3}\cos(x/3). \end{aligned}$$

Substituting ψ, ϕ into the general solution which was obtained in (b), we get

$$u(x,y) = \frac{1}{324}(3x+y)y^4 - \frac{1}{540}y^5 + y\left[\cos(x+y/3) - \frac{1}{3}\cos(x+y/3)\right] + \sin(x+y/3).$$

3.3 (a) Compute $\Delta = 4 > 0$. Therefore the equation is hyperbolic. We need to solve $v_x^2 + 4v_xv_y = 0$. This leads to two equations: $v_x = 0$ which implies $s(x,y) = y$, and $v_x + 4v_y = 0$ which implies $t(x,y) = y - 4x$. Writing $w(s,t) = u(x,y)$, the equation is transformed into $w_{st} + \frac{1}{4}w_t = 0$.
(b) Using $W := w_t$, the general solution is found to be $u(x,y) = f(y-4x)\mathrm{e}^{-y/4} + g(y)$, for arbitrary functions $f, g \in C^2(\mathbb{R})$.
(c) $u(x,y) = (-y/2 + 4x)\mathrm{e}^{-y/4}$.

3.5 (a) The equation's coefficients are $a = x$, $2b = 0$, $c = -y$. Thus, $b^2 - ac = xy$, implying that the equation is hyperbolic when $xy > 0$, elliptic when $xy < 0$, and parabolic when $xy = 0$ (but this is not a domain!).
(b) The characteristic equation is $xy'^2 - y = 0$, or $y'^2 = y/x$.
(1) When $xy > 0$ there are two real roots $y' = \pm\sqrt{y/x}$. Suppose for instance that $x, y > 0$. Then the solution is $\sqrt{y} \pm \sqrt{x} =$ constant. We define the new variables $s(x,y) = \sqrt{y} + \sqrt{x}$ and $t(x,y) = \sqrt{y} - \sqrt{x}$.
(2) When $xy < 0$ there are two complex roots $y' = \pm\mathrm{i}\sqrt{|y/x|}$. We choose $y' = \mathrm{i}\sqrt{|y/x|}$. The solution of the ODE is $2\mathrm{sign}(y)\sqrt{|y|} = \mathrm{i}2\mathrm{sign}(x)\sqrt{|x|} +$ constant.

Divide by $2\mathrm{sign}(y) = -2\mathrm{sign}(x)$ to obtain $\sqrt{|y|} + \mathrm{i}\sqrt{|x|} = \text{constant}$. We thus define the new variables $s(x,y) = \sqrt{|x|}$ and $t(x,y) = \sqrt{|y|}$.

3.7 (a) Here $a = 1$, $2b = 2$, $c = 1 - q$; thus $b^2 - ac = q$, and, therefore:
The equation is hyperbolic for $q > 0$, i.e. for $y > 1$.
The equation is elliptic for $q < 0$, i.e. for $y < -1$.
The equation is parabolic for $q = 0$, i.e. for $|y| \le 1$.
(b) The characteristics equation is $(y')^2 - 2y' + (1-q) = 0$; its roots are $y'_{1,2} = 1 \pm \sqrt{q}$.
(1) The hyperbolic regime $y > 1$: We have two real roots $y'_{1,2} = 1 \pm 1$. The solutions of the ODEs are

$$y_1 = \text{constant}, \quad y_2 = 2x + \text{constant}.$$

Hence the new variables are $s(x,y) = y$ and $t(x,y) = y - 2x$.
(2) The elliptic regime $y < -1$: The two roots are imaginary: $y'_{1,2} = 1 \pm \mathrm{i}$. Choosing one of them $y' = 1 + \mathrm{i}$, we obtain $y = (1+\mathrm{i})x + \text{constant}$. The new variables are $s(x,y) = y - x$, $t(x,y) = x$.
(3) The parabolic regime $|y| \le 1$: There is a single real root $y' = 1$; The solution of the resulting ODE is $y = x + \text{constant}$. The new variables are $s(x,y) = x,\ t(x,y) = x - y$.

3.11 (a) The general solution is given by $v(s,t) = f(s) + g(t)$, or

$$u(x,y) = F(\cos x + x - y) + G(\cos x - x - y). \tag{12.8}$$

The first condition implies

$$f(y) = u(0,y) = F(1-y) + G(1-y), \tag{12.9}$$

while the second condition gives

$$g(y) = u_x(0,y) = F'(1-y) - G'(1-y). \tag{12.10}$$

Integrating both sides of (12.10) we get

$$\int_0^y g(s)\mathrm{d}s = -F(1-y) + F(1) + G(1-y) - G(1). \tag{12.11}$$

By summing up equations (12.9) and (12.11) we obtain

$$\int_0^y g(s)\mathrm{d}s + f(y) = 2G(1-y) + F(1) - G(1),$$

that is, $G(x) = \frac{1}{2}\left[\int_0^{1-x} g(s)\mathrm{d}s + f(1-x) - F(1) + G(1)\right]$. This implies

$$F(x) = f(1-x) - \frac{1}{2}\left[\int_0^{1-x} g(s)\mathrm{d}s + f(1-x) - F(1) + G(1)\right].$$

Therefore,

$$u(x,y) = \frac{1}{2}\left[f(1-\cos x - x + y) + f(1 - \cos x + x + y)\right] + \frac{1}{2}\left[\int_{1-\cos x - x + y}^{1 - \cos x + x + y} g(s)\,\mathrm{d}s\right].$$

(b) The solution is classic if it is twice differentiable. Thus, one should require that f would be twice differentiable, and that g would be differentiable.

Chapter 4

4.3 (a)

$$u(x,1) = \frac{f(x+2)+f(x-2)}{2} + \frac{1}{4}\int_{x-2}^{x+2} g(s)\mathrm{d}s.$$

$$u(x,1) \quad = \quad \begin{cases} 0 & x < -3, \\ \frac{1}{2}[1-(x+2)^2] & -3 \le x \le -1, \\ x+1 & -1 \le x \le 0, \\ 1 & 0 \le x \le 1, \\ \frac{1}{2}[1-(x-2)^2]+1 & 1 \le x \le 3, \\ 4-x & 3 \le x \le 4, \\ 0 & x > 4. \end{cases}$$

(b) $\lim_{t\to\infty} u(5,t) = 1$.
(c) The solution is singular at the lines: $x \pm 2t = \pm 1, 2$.
(d) The solution is continuous at all points.

4.5 (a) Using d'Alembert's formula:

$$u(x,t) = \frac{1}{2}\left[u_0(x-t)+u_0(x+t)\right] + \frac{1}{2}\left[U_0(x+t)-U_0(x-t)\right],$$

where $u_0(x) = u(x,0) = f(x)$, $U_0(x) = \int_0^x u_t(s,0)\,\mathrm{d}s = \int_0^x g(s)\,\mathrm{d}s$. Therefore, the backward wave is

$$u_r(x,t) = \frac{1}{2}\left[u_0(x+t)+U_0(x+t)\right],$$

and the forward wave is

$$u_p(x,t) = \frac{1}{2}\left[u_0(x-t)-U_0(x-t)\right].$$

Hence

$$u_r(x,t) = \begin{cases} 12(x+t)-(x+t)^2 & 0 \le x+t \le 4, \\ 0 & x+t < 0, \\ 32 & x+t > 4. \end{cases}$$

Similarly:

$$u_p(x,t) = \begin{cases} -4(x-t)-(x-t)^2 & 0 \le x-t \le 4, \\ 0 & x-t < 0, \\ -32 & x-t > 4. \end{cases}$$

(d) The explicit representation formulas for the backward and forward waves of (a) imply that the limit is 32, since for t large enough we have $5+t > 4$ *and* $5-t < 0$.

4.7 (a) Consider a forward wave $u = u_p(x,t) = \psi(x-t)$. Then

$$\begin{aligned} u_p(x_0-a, t_0-b)+u_p(x_0+a, t_0+b) &= \psi(x_0-t_0-a+b)+\psi(x_0-t_0+a-b) \\ &= u_p(x_0-b, t_0-a)+u_p(x_0+b, t_0+a). \end{aligned}$$

Similarly, we obtain the equality for a backward wave $u = u_r(x,t) = \phi(x+t)$. Since every solution of the wave equation is a linear combination of forward and backward waves, the statement follows.
(b) $u(x_0 - ca, t_0 - b) + u(x_0 + ca, t_0 + b) = u(x_0 - cb, t_0 - a) + u(x_0 + cb, t_0 + a)$.
(c)

$$u(x,t) = \begin{cases} \frac{f(x+t)+f(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,\mathrm{d}s & t \le x, \\ \frac{f(x+t)-f(t-x)}{2} + \frac{1}{2}\int_{t-x}^{x+t} g(s)\,\mathrm{d}s + h(t-x) & t \ge x. \end{cases}$$

(d) The corresponding compatibility conditions are $h(0) = f(0)$, $h'(0) = g(0)$, $h''(0) = f''(0)$. If these conditions are not satisfied the solution is singular along the line $x - t = 0$.
(e)

$$u(x,t) = \begin{cases} \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s & ct \le x, \\ \frac{f(x+ct)-f(ct-x)}{2} + \frac{1}{2c}\int_{ct-x}^{x+ct} g(s)\,\mathrm{d}s + h(t-\frac{x}{c}) & ct \ge x. \end{cases}$$

The corresponding compatibility conditions are $h(0) = f(0)$, $h'(0) = g(0)$, $h''(0) = c^2 f''(0)$. If these conditions are not satisfied the solution is singular along the line $x - ct = 0$.

4.9 To obtain a homogeneous equation, we use the substitution $v(x,t) = u(x,t) - t^2/2$. The initial condition is unchanged. We conclude that v solves the problem

$$v_{tt} - v_{xx} = 0, \quad v(x,0) = x^2, \quad v_t(x,0) = 1.$$

Using d'Alembert's formula we get

$$v(x,t) = \frac{1}{2}\left[(x+t)^2 + (x-t)^2\right] + t = x^2 + t^2 + t,$$

that is, $u(x,t) = x^2 + t + 3t^2/2$.

4.11 d'Alembert's formula implies

$$P(x,t) = \frac{1}{2}\left[f(x+4t) + f(x-4t)\right] + \frac{1}{8}\left[H(x+4t) - H(x-4t)\right],$$

where $H(x) = \int_0^x g(s)\,\mathrm{d}s$. We get

$$H(x) = \begin{cases} x & |x| \le 1, \\ 1 & x > 1, \\ -1 & x < -1. \end{cases} \tag{12.12}$$

Let us look at the solution at the point $x_0 = 10$; notice that

$$f(10+4t) = 0, \quad f(10-4t) \leq 10, \quad |H(t)| \leq 1, \quad t > 0.$$

Therefore,

$$P(10,t) \leq 5 + \frac{1}{4} = \frac{21}{4} < 6,$$

and the structure will not collapse.

4.13 We use the transformation

$$v(x,t) = u(x,t) - \mathrm{e}^x$$

to obtain for v a homogeneous problem:

$$v_{tt} - 4u_{xx} = 0, \quad v(x,0) = f(x) - \mathrm{e}^x, \quad v_t(x,0) = g(x).$$

d'Alembert's formula implies

$$v(x,t) = \frac{1}{2}\left[f(x+2t) - \mathrm{e}^{x+2t} + f(x-2t) + \mathrm{e}^{x-2t}\right] + \frac{1}{4}\left[H(x+2t) - H(x-2t)\right],$$

where $H(x) = \int_0^x g(s)\,\mathrm{d}s$. Thus,

$$H(x) = \begin{cases} x - x^3/3 & |x| \leq 1 \\ 2/3 & x > 1 \\ -2/3 & x < -1. \end{cases} \tag{12.13}$$

Returning to u:

$$u(x,t) = \frac{1}{2}\left[f(x+2t) - \mathrm{e}^{x+2t} + f(x-2t) - \mathrm{e}^{x-2t}\right] + \frac{1}{4}\left[H(x+2t) - H(x-2t)\right] - \mathrm{e}^x.$$

(a) The solution is not classical when $x \pm 2t = -1, 0, 1, 2, 3$.
(b) $u(1,1) = 1/3 + e - \mathrm{e}^3/2 - \mathrm{e}^{-1}/2$.

4.15 Denote $v = u_x$. We obtain for $v(x,t)$ the following Cauchy problem:

$$v_{tt} - v_{xx} = 0, \quad v(x,0) = 0, \quad v_t(x,0) = \sin x.$$

Therefore,

$$v(x,t) = \frac{1}{2}\int_{x-t}^{x+t} \sin s\,\mathrm{d}s = \frac{1}{2}\left[\cos(x-t) - \cos(x+t)\right],$$

and the solution is

$$u(x,t) = \int v(x,t)\,\mathrm{d}x + f(t) = \frac{1}{2}\left[\sin(x-t) - \sin(x+t)\right] + f(t),$$

where $f(t)$ is an arbitrary function.

4.17 (a) Change variables to obtain the canonical form of the wave equation:

$$\zeta = \frac{x+t}{2} \qquad \eta = \frac{t-x}{2}.$$

We get

$$u_{\zeta\eta} = \cos 2\zeta.$$

The general solution is given by

$$u(\zeta,\eta) = \frac{\eta}{2}\sin 2\zeta + \psi_1(\eta) + \psi_2(\zeta),$$

where ψ_1, ψ_2 are arbitrary functions. Returning to the original variables we find

$$u(x,t) = \frac{t-x}{4}\,\sin(x+t) + \phi_1(x+t) + \phi_2(x-t).$$

To find the required solution we substitute the initial conditions into the above solution:

$$u(x,0) = -\frac{x}{4}\,\sin x + \phi_1(x) + \phi_2(x) = x,$$

$$u_t(x,0) = \frac{1}{4}\,\sin x - \frac{x}{4}\,\cos x + {\phi_1}'(x) - {\phi_2}'(x) = \sin x.$$

Integrating the last equation:

$$\phi_1(x) - \phi_2(x) - \frac{1}{2}\,\cos x - \frac{x}{4}\sin x = -\cos x.$$

Eliminating ϕ_1, ϕ_2 yields

$$\phi_1(x) = \frac{x}{2} + \frac{x}{4}\,\sin x - \frac{\cos x}{4}, \quad \text{and} \quad \phi_2(x) = \frac{x}{2} + \frac{\cos x}{4},$$

which implies

$$u(x,t) = x + \frac{t}{2}\,\sin(x+t) + \frac{\cos(x-t)}{4} - \frac{\cos(x+t)}{4}.$$

(b) Similarly, we obtain the equations

$$-\frac{x}{4}\,\sin x + \phi_1(x) + \phi_2(x) = 0,$$

$$-\frac{1}{2}\,\cos x - \frac{x}{4}\,\sin x + \phi_1(x) - \phi_2(x) = 0,$$

which imply that

$$\phi_1(x) = \frac{x}{4}\,\sin x + \frac{1}{4}\,\cos x, \qquad \phi_2(x) = -\frac{1}{4}\,\cos x.$$

Solving the equation together with the initial conditions gives

$$v(x,t) = \frac{t}{2}\,\sin(x+t) + \frac{1}{4}\,\cos(x+t) - \frac{1}{4}\,\cos(x-t).$$

(c) The function $w(x,t) = \frac{1}{2}\cos(x+t) - \frac{1}{2}\cos(x-t) - x$ solves the homogeneous wave equation $w_{tt} - w_{xx} = 0$, and satisfies the initial conditions $w(x,0) = x$, $w_t(x,0) = \sin x$.
(d) w is an odd function of x.

4.19 The general solution of the wave equation is

$$u(x,t) = F(x+t) + G(x-t).$$

Hence,

$$u_x(x,t) = F'(x+t) + G'(x-t).$$

Substituting $x - t = 1$ into the above expression implies

$$u_x(x,t)\big|_{x-t=1} = F'(2t+1) + G'(1) = \text{constant}.$$

Thus, $F'(s) = \text{constant}$, implying $F(s) = ks$. We are also given that

$$1 = u(x,0) = F(x) + G(x) = kx + G(x).$$

Therefore, $G(x) = 1 - kx$. On the other hand,

$$3 = u(1,1) = F(2) + G(0) = 2k + (1 - 0 \times k),$$

i.e. $k = 1$. We conclude

$$F(x) = x, \quad G(x) = 1 - x, \quad u(x,t) = 1 + 2t.$$

Thanks to the method in which the solution was constructed we can infer that it is unique.

Chapter 5

5.1 The solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} B_n \mathrm{e}^{-17n^2 t} \sin nx. \tag{12.14}$$

Substituting the initial conditions into (12.14) gives

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin nx = f(x).$$

To find the coefficients B_n we expand $f(x)$ into a series in the eigenfunctions:

$$B_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,\mathrm{d}x = \frac{2}{\pi}\int_{\pi/2}^{\pi} 2\sin nx\,\mathrm{d}x = \frac{4}{\pi n}\left[\cos\left(\frac{n\pi}{2}\right) - (-1)^n\right].$$

It follows that the solution is

$$u(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\cos\left(\frac{n\pi}{2}\right) - (-1)^n\right]\mathrm{e}^{-17n^2 t}\sin nx.$$

5.2 Using trigonometric identities we express the solution in the form

$$u(x,t) = u_1(x,t) + u_2(x,t) + \frac{A_0}{2},$$

where u_1 is a forward wave, and u_2 is a backward wave (the constant $A_0/2$ can be considered either a forward wave or a backward wave):

$$u_1(x,t) = -\frac{B_0}{4c}(x-ct) + \sum_{n=1}^{\infty}\left\{\frac{A_n}{2}\cos\left[\frac{n\pi(x-ct)}{L}\right] - \frac{B_n}{2}\sin\left[\frac{n\pi(x-ct)}{L}\right]\right\},$$

$$u_2(x,t) = \frac{B_0}{4c}(x+ct) + \sum_{n=1}^{\infty}\left\{\frac{A_n}{2}\cos\left[\frac{n\pi(x+ct)}{L}\right] + \frac{B_n}{2}\sin\left[\frac{n\pi(x+ct)}{L}\right]\right\}.$$

5.3 (a) Separating variables we infer that there is a constant, denoted by λ such that

$$\frac{T_{tt}}{c^2 T} = \frac{X_{xx}}{X} = -\lambda. \tag{12.15}$$

Equation (12.15) leads to the coupled ODE system

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = -\lambda X \qquad 0 < x < L, \tag{12.16}$$

$$\frac{\mathrm{d}^2 T}{\mathrm{d}t^2} = -\lambda c^2 T \qquad t > 0. \tag{12.17}$$

Since u is not the trivial solution, the boundary conditions imply $X(0) = X(L) = 0$. Thus, the function X must satisfy the eigenvalue problem

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + \lambda X = 0 \qquad 0 < x < L, \tag{12.18}$$

$$X(0) = X(L) = 0. \tag{12.19}$$

We already saw that the solution to the problem (12.18)–(12.19) is the infinite sequence

$$X_n(x) = \sin \frac{n\pi x}{L}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad n = 1, 2, \ldots.$$

We proceed to equation (12.17). Using the eigenvalues obtained above we find

$$T_n(t) = \gamma_n \sin(\sqrt{\lambda_n c^2}\, t) + \delta_n \cos(\sqrt{\lambda_n c^2}\, t) \quad n = 1, 2, 3, \ldots. \tag{12.20}$$

We have thus derived the separated solutions

$$u_n(x,t) = X_n(x) T_n(t) = \sin \frac{n\pi x}{L} \left(A_n \cos \frac{c\pi nt}{L} + B_n \sin \frac{c\pi nt}{L} \right) \quad n = 1, 2, 3, \ldots.$$

Superposing these solutions we write

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{c\pi nt}{L} + B_n \sin \frac{c\pi nt}{L} \right) \sin \frac{n\pi x}{L} \tag{12.21}$$

as the (generalized) solution to the problem of string vibrations with Dirichlet boundary conditions. It remains to find the coefficients A_n, B_n. For this purpose we use the initial conditions

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x, \quad B_n = \frac{2}{cn\pi} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x \quad n \geq 1.$$

5.4 We substitute the initial conditions into the general solution (12.21), where $L = \pi$ and $c = 1$:

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos nt + B_n \sin nt) \sin nx. \tag{12.22}$$

We get

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin nx = \sin^3 x = -\frac{1}{4} \sin 3x + \frac{3}{4} \sin x, \tag{12.23}$$

$$\frac{\partial u(x,0)}{\partial t} = \sum_{n=1}^{\infty} n B_n \sin nx = \sin 2x. \tag{12.24}$$

Hence,

$$A_1 = -1/4, \quad A_3 = 3/4, \quad B_2 = 1/2,$$

and $A_n = 0$ if $n \neq 1, 3$, $B_n = 0$ if $n \neq 2$. We conclude that the formal solution is

$$u(x,t) = -\frac{1}{4}\sin 3x \cos 3t + \frac{3}{4}\sin x \cos t + \frac{1}{2}\sin 2x \sin 2t.$$

This is a finite sum of smooth functions and therefore is a classical solution.

5.5 (a) The eigenfunctions and eigenvalues of the relevant Sturm–Liouville system are

$$X_n(x) = \cos\left(\frac{n\pi x}{L}\right), \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad n = 0, 1, 2, \dots .$$

Therefore, the solution has the form

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \mathrm{e}^{-k\pi^2 n^2 t/L^2} \cos\left(\frac{n\pi x}{L}\right),$$

where

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x \qquad n \geq 0.$$

(c) The obtained function is a classical solution of the equation for all $t > 0$, since if f is continuous then the exponential decay implies that for every $\varepsilon > 0$ the series and all its derivatives converge uniformly for all $t > \varepsilon > 0$. For the same reason, the series (without $A_0/2$) converges uniformly to zero (as a function of x) in the limit $t \to \infty$. Thus,

$$\lim_{t\to\infty} u(x,t) = \frac{A_0}{2}.$$

It is instructive to compute A_0 by an alternative method. Notice that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L u(x,t)\,\mathrm{d}x = \int_0^L u_t(x,t)\,\mathrm{d}x = k\int_0^L u_{xx}(x,t)\,\mathrm{d}x$$
$$= k\left[u_x(L,t) - u_x(0,t)\right] = 0,$$

where the last equality follows from the Neumann boundary condition. Hence,

$$\int_0^L u(x,t)\,\mathrm{d}x = \int_0^L u(x,0)\,\mathrm{d}x = \int_0^L f(x)\,\mathrm{d}x$$

holds for all $t > 0$. Since the uniform convergence of the series implies the convergence of the integral series, we infer

$$\frac{A_0}{2} = \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x\ .$$

A physical interpretation: We have shown that the quantity $\int_0^L u(x,t)\,\mathrm{d}x$ is conserved in a one-dimensional insulated rod. The quantity $ku_x(x,t)$ measures the heat flux at a point x and time t. The homogeneous Neumann condition amounts to stating that there is zero flux at the rod's ends. Since there are no heat sources either (the equation is homogeneous), the temperature's gradient decays; therefore the temperature converges to a constant, such that the total stored energy is the same as the initial energy.

5.7 To obtain a homogeneous equation write $u = v + w$ where $w = w(t)$ satisfies

$$w_t - kw_{xx} = A\cos\alpha t, \qquad w(x,0) \equiv 0.$$

Therefore,

$$w(t) = \frac{A}{\alpha}\sin\alpha t\ .$$

Note that w satisfies also $w_x(0,t) = w_x(1,t) = 0$. Therefore, v should solve

$$\begin{aligned} v_t - kv_{xx} &= 0 && 0 < x < 1, \quad t > 0, \\ v_x(0,t) = v_x(1,t) &= 0 && t \geq 0, \\ v(x,0) &= 1 + \cos^2 \pi x && 0 \leq x \leq 1. \end{aligned}$$

Thus,

$$v(x,t) = \sum_{n=0}^{\infty} B_n \mathrm{e}^{-kn^2\pi^2 t} \cos n\pi x = B_0 + \sum_{n=1}^{\infty} B_n \mathrm{e}^{-kn^2\pi^2 t} \cos n\pi x.$$

The coefficients B_n are found to be

$$B_0 = \int_0^1 \left[1 + \cos^2(\pi x)\right] \mathrm{d}x = \frac{3}{2}, \quad B_n = 2\int_0^1 \left[1 + \cos^2(\pi x)\right] \cos n\pi x \,\mathrm{d}x \quad n \geq 1.$$

We obtain

$$B_2 = \int_0^1 \left(\frac{3}{2} + \cos 2\pi x\right) \cos 2\pi x \,\mathrm{d}x = 1/2, \; B_n = \int_0^1 \left(\frac{3}{2} + \cos 2\pi x\right) \cos n\pi x \,\mathrm{d}x = 0, \; n \neq 0, 2.$$

Finally,

$$u(x,t) = 3/2 + 1/2\cos 2\pi x \mathrm{e}^{-4k\pi^2 t} + \frac{A}{\alpha}\sin\alpha t\ .$$

Compare this problem with Example 6.45 and the discussion therein.

5.9 (a) The associated eigenvalue problem is

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + hX + \lambda X = 0, \quad X(0) = X(\pi) = 0,$$

while the ODE for $T(t)$ is

$$\frac{\mathrm{d}^2 T}{\mathrm{d}t} + \lambda T = 0.$$

The solutions are

$$X_n(x) = B_n \sin nx, \qquad \lambda_n = n^2 - h \qquad n \geq 1,$$

$$T_n(t) = \mathrm{e}^{(-n^2+h)t}.$$

Hence the problem's solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \mathrm{e}^{(-n^2+h)t} \sin nx,$$

where

$$B_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx \,\mathrm{d}x = -\frac{4[(-1)^n - 1]}{\pi n^3}.$$

(b) $\lim_{t\to\infty} u(x,t)$ exists if and only if $h \leq 1$. When $h < 1$ the series converges uniformly to 0. If $h = 1$, the series converges to $B_1 \sin x$ which is the principal eigenfunction (see Definition 6.36 and the discussion therein).

5.10 (a) The solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin n\pi x \mathrm{e}^{-(n^2\pi^2-\alpha)t}.$$

The coefficients A_n are given by expanding $f(x) = x$ into a generalized Fourier series in the functions $\sin n\pi x$.
(c) Let us rewrite the solution in the form

$$u(x,t) = A_1 \sin \pi x \mathrm{e}^{-(\pi^2-\alpha)t} + \sum_{n=2}^{\infty} A_n \sin n\pi x \mathrm{e}^{-(n^2\pi^2-\alpha)t}.$$

The condition on α implies that the infinite series decays as $t \to \infty$. In addition, because $\alpha > \pi^2$, it follows that a necessary and sufficient condition for the limit to exist is $A_1 = 0$.

5.11 (a) The domain of dependence is the interval $[1/3 - 1/10, 1/3 + 1/10]$ along the x axis.
(b) Part (a) implies that the domain of dependence does *not* include the boundary. Therefore, we can use d'Alembert's formula, and consider the initial conditions as if they were given on the entire real line, and not on a finite interval. We obtain at once

$$u(3^{-1}, 10^{-1}) = -\frac{1}{2} \times \frac{65}{15^3} = -\frac{13}{1350}.$$

(c) The formal solution is

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos n\pi x \cos n\pi t.$$

Substituting the initial data into the proposed solution yields

$$\sum_{n=0}^{\infty} A_n \cos n\pi x = 2\sin^2(2\pi x) = 1 - \cos 4\pi x.$$

Therefore,

$$A_0 = 1, \quad A_4 = -1, \quad A_n = 0 \quad \forall n \neq 1, 4.$$

We conclude that the solution is given by

$$u(x,t) = 1 - \cos 4\pi x \cos 4\pi t.$$

5.13 The eigenvalue problem is

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + (\lambda - 1)X = 0, \qquad X(0) = X'(1) = 0,$$

while the ODE for $T(t)$ is

$$\frac{\mathrm{d}T}{\mathrm{d}t} + \lambda T = 0.$$

Thus,

$$\lambda_n = \frac{(2n+1)^2\pi^2}{4} + 1, \qquad X_n(x) = B_n \sin\left(\frac{2n+1}{2}\pi x\right) \qquad n = 0, 1, 2, \ldots.$$

This leads to a solution of the form

$$u(x,t) = \mathrm{e}^{-t} \sum_{n=0}^{\infty} B_n \mathrm{e}^{-(2n+1)^2 t\pi^2/4} \sin\left(\frac{2n+1}{2}\pi x\right).$$

Computing B_n explicitly we get

$$B_n = 2\int_0^1 x(2-x)\sin\left(\frac{2n+1}{2}\pi x\right)\,\mathrm{d}x = \frac{32}{(2n+1)^3\pi^3}.$$

This solution is clearly classical.

5.14 Let us compute

$$\frac{\partial u}{\partial t} = v(x,t,t) + \int_0^t v_t(x,t,s)\,\mathrm{d}s = v(x,t,t) + \int_0^t v_{xx}(x,t,s)\,\mathrm{d}s,$$

$$\frac{\partial^2 u}{\partial x^2} = \int_0^t v_{xx}(x,t,s)\,\mathrm{d}s,$$

(use Formula (5) of Section A.2). Therefore,

$$u_t - u_{xx} = F(x,t).$$

The initial and boundary conditions for u are obtained at once from those of v.

5.15 Let u_1, u_2 be a pair of solutions for the system. Set $v = u_1 - u_2$. We need to show that $v \equiv 0$. Thanks to the superposition principle, the function v solves the homogeneous system

$$\begin{aligned} v_{tt} - c^2 v_{xx} &= 0 \qquad && 0 < x < L,\ t > 0, \\ v_x(0,t) = 0, \quad v(L,t) &= 0 && t \geq 0, \\ v(x,0) = v_t(x,0) &= 0 && 0 \leq x \leq L. \end{aligned}$$

Define now

$$E(t) = \frac{1}{2}\int_0^L \left(v_t^2 + c^2 v_x^2\right) \mathrm{d}x.$$

From the homogeneous initial conditions $E(0) = 0$. We proceed to compute:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_0^L \left(v_t v_{tt} + c^2 v_x v_{xt}\right) \mathrm{d}x.$$

Integrating by parts and using the boundary conditions we compute

$$\int_0^L v_x v_{xt}\,\mathrm{d}x = -\int_0^L v_t v_{xx}\,\mathrm{d}x + v_t(L,t)v_x(L,t) - v_t(0,t)v_x(0,t) = -\int_0^L v_t v_{xx}\,\mathrm{d}x,$$

hence

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_0^L v_t\left(v_{tt} - c^2 v_{xx}\right)\,\mathrm{d}x = 0.$$

This gives $E(t) = E(0) = 0$ for all $0 \le t < \infty$. Therefore, $v_t = v_x \equiv 0$, i.e. $v(x,t) = \text{constant}$; but $v(x,0) = 0$, implying $v(x,t) \equiv 0$.

5.17 Let u_1 and u_2 be a pair of solutions. Set $v = u_1 - u_2$. We need to show that $v \equiv 0$. Thanks to the superposition principle v solves the homogeneous system

$$\begin{aligned} v_{tt} - c^2 v_{xx} + hv &= 0 \qquad & -\infty < x < \infty,\ t > 0, \\ \lim_{x\to\pm\infty} v(x,t) = \lim_{x\to\pm\infty} v_x(x,t) = \lim_{x\to\pm\infty} v_t(x,t) &= 0, & t \ge 0, \\ v(x,0) = v_t(x,0) &= 0 & -\infty < x < \infty. \end{aligned}$$

Let $E(t)$ be as suggested in the problem. The initial conditions imply $E(0) = 0$. Differentiating formally $E(t)$ by t we write

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_{-\infty}^{\infty} \left(v_t v_{tt} + c^2 v_x v_{xt} + hvv_t\right) \mathrm{d}x,$$

assuming that all the integrals converge (we ought to be careful since the integration is over the entire real line).

We compute

$$\int_{-\infty}^{\infty} v_x v_{xt}\,\mathrm{d}x = -\int_{-\infty}^{\infty} v_t v_{xx}\,\mathrm{d}x + \int_{-\infty}^{\infty} \frac{\partial(v_x v_t)}{\partial x}\,\mathrm{d}x.$$

Using the homogeneous boundary conditions

$$\int_{-\infty}^{\infty} \frac{\partial(v_x v_t)}{\partial x}\,\mathrm{d}x = \lim_{x\to\infty} v_x(x,t)v_t(x,t) - \lim_{x\to-\infty} v_x(x,t)v_t(x,t) = 0,$$

hence, $\int_{-\infty}^{\infty} v_x v_{xt}\mathrm{d}x = -\int_{-\infty}^{\infty} v_{xx} v_t \mathrm{d}x$. Conclusion:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_{-\infty}^{\infty} v_t\left(v_{tt} - c^2 v_{xx} + hv\right)\mathrm{d}x = 0\ .$$

We verified that $E(t) = E(0) = 0$ for all t. The positivity of h implies that $v \equiv 0$.

5.18 Let $v = u_1 - u_2$ where u_1, u_2 are two solutions. Clearly v satisfies

$$\begin{aligned} v_t - kv_{xx} &= 0 \qquad && 0 < x < L,\ t > 0, \\ v(0,t) - \alpha v_x(0,t) = 0, \quad v(L,t) + \beta v_x(L,t) &= 0 && t \geq 0, \\ v(x,0) &= 0 && 0 < x < L. \end{aligned}$$

Set

$$E(t) = \frac{1}{2}\int_0^L v^2(x,t)\,\mathrm{d}x.$$

The equation $v_t = kv_{xx}$ gives

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_0^L v(x,t)v_t(x,t)\,\mathrm{d}x = k\int_0^L v(x,t)v_{xx}(x,t)\,\mathrm{d}x = -k\int_0^L v_x^2(x,t)\,\mathrm{d}x$$
$$+k\left[v(L,t)v_x(L,t) - v(0,t)v_x(0,t)\right].$$

From the boundary conditions, $v(0,t) = \alpha v_x(0,t)$, $v(L,t) = -\beta v_x(L,t)$. Therefore,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -k\int_0^L v_x^2(x,t)\,\mathrm{d}x - k\alpha v_x^2(0,t) - k\beta v_x^2(L,t) \leq 0.$$

Therefore, $E(t) \leq E(0)$ for all $t \geq 0$. Since $E(t) \geq 0$ and $E(0) = 0$, we obtain $E(t) = 0$ for all $t \geq 0$, and thus $v \equiv 0$.

5.19 (b) We consider the homogeneous equation

$$\begin{aligned} (y^2 v_x)_x + (x^2 v_y)_y &= 0 \qquad && (x,y) \in D, \\ v(x,y) &= 0 && (x,y) \in \Gamma. \end{aligned}$$

Multiply the equation by v and integrate over D:

$$\int\int_D v\left[(y^2v_x)_x + (x^2v_y)_y\right]\mathrm{d}x\mathrm{d}y = 0.$$

Using the identity of part (a) we obtain

$$\int\int_D v\left[(y^2v_x)_x + (x^2v_y)_y\right]\mathrm{d}x\mathrm{d}y = -\int\int_D \left[(yv_x)^2 + (xv_y)^2\right]\mathrm{d}x\mathrm{d}y$$
$$+\int\int_D \mathrm{div}\,\left(y^2vv_x, x^2vv_y\right)\mathrm{d}x\mathrm{d}y.$$

Using further the divergence theorem (see Formula (2) in Section A.2):

$$\int\int_D \mathrm{div}\,\left(vy^2v_x, x^2vv_y\right)\mathrm{d}x\mathrm{d}y = \int_\Gamma \left[vy^2v_x\mathrm{d}y - vx^2v_y\mathrm{d}x\right] = 0,$$

where in the last equality we used the homogeneous boundary condition $v \equiv 0$ on Γ. We infer that the energy integral satisfies

$$E[v] := \int\int_D \left[(yv_x)^2 + (xv_y)^2\right]\,\mathrm{d}x\mathrm{d}y = 0,$$

hence $v_x = v_y = 0$ in D. We conclude that $v(x,y)$ is constant in D, and then the homogeneous boundary condition implies that this constant must vanish.

Chapter 6

6.1 (a) It is easy to check that 0 is not an eigenvalue. Assume there exists an eigenvalue $\lambda < 0$. Multiply the equation by the associated eigenfunction u and integrate to obtain

$$\int_0^1 u u_{xx}\, \mathrm{d}x + \lambda \int_0^1 u^2\, \mathrm{d}x = 0.$$

Integrating further by parts:

$$0 = -\int_0^1 u_x^2\, \mathrm{d}x + \lambda \int_0^1 u^2\, \mathrm{d}x + u_x(1)u(1) - u_x(0)u(0).$$

Using the boundary conditions one can deduce $u_x(1)u(1) - u_x(0)u(0) = -u(0)^2 - u(1)^2 \leq 0$. We reached a contradiction to our assumption $\lambda < 0$.
(b) Using part (a) we set $\lambda = \mu^2$ (say, for positive μ). The general solution to the ODE is given by

$$u(x) = A \sin \mu x + B \cos \mu x.$$

The boundary conditions dictate

$$u(0) = B = u'(0) = \mu A, \qquad u(1) = A \sin \mu + B \cos \mu = -u'(0) = -\mu A \cos \mu + \mu B \sin \mu.$$

We obtain the transcendental equation

$$\frac{2\mu}{\mu^2 - 1} = \tan \mu\ .$$

To obtain a better feeling for the solutions of this equation, we can draw the graphs of the functions $\frac{2\mu}{\mu^2-1}$ and $\tan \mu$. The roots μ_i are determined by the intersection points of these graphs, and the eigenvalues are $\lambda_i = \mu_i^2$.
(c) Taking the limit $\lambda \to \infty$ (or $\mu \to \infty$), it follows that μ_n satisfies the asymptotic relation $\mu_n \sim n\pi$, where $n\pi$ is the root of the n-th branch of $\tan \mu$. Therefore, $\lambda_n \approx n^2\pi^2$ as $n \to \infty$.

6.2 (a) Since all the eigenvalues can be seen to be positive, we set $\lambda = \mu^2 > 0$. Using Formula (3) of Section A.3, it follows that the general solution of the corresponding ODE is given by

$$u(x) = a \sin(|\mu| \ln x) + b \cos(|\mu| \ln x),$$

and the boundary condition implies

$$u(1) = b = u'(\mathrm{e}) = a|\mu| \cos(|\mu|) = 0.$$

We conclude that $|\mu| = (n + 1/2)\pi$,

$$u_n(x) = \sin\left[\frac{(2n+1)\pi}{2} \ln x\right], \qquad \lambda_n = \left[\frac{(2n+1)\pi}{2}\right]^2 \quad n = 0, 1, \ldots.$$

(b) It is convenient to use the variable $t = \ln x$. The inner product becomes

$$\int_1^{\mathrm{e}} \frac{1}{x} \sin\left[\frac{(2n+1)\pi}{2} \ln x\right] \sin\left[\frac{(2m+1)\pi}{2} \ln x\right] \mathrm{d}x$$

$$= \int_0^1 \sin\left[\frac{(2n+1)\pi}{2} t\right] \sin\left[\frac{(2m+1)\pi}{2} t\right] \mathrm{d}t = 0 \qquad n \neq m.$$

6.3 (a) We examine whether the function

$$v(x) = x^{-1/2} \sin(\alpha \ln x)$$

indeed satisfies the ODE:

$$(x^2\, v')' + \lambda\, v = -\frac{(1 + 4\alpha^2 - 4\lambda) \sin(\alpha \ln x)}{4\sqrt{x}} = 0,$$

and in order for the ODE to hold, we require

$$1 + 4\alpha^2 - 4\lambda = 0 \quad \Longrightarrow \alpha = \pm\sqrt{\lambda - 1/4}\,, \quad \lambda > 1/4\,.$$

Thus, the function

$$v(x) = x^{-1/2} \sin(\sqrt{\lambda - 1/4}\, \ln x)$$

indeed solves the equation. This function vanishes at $x = 1$ since $\ln 1 = 0$. To determine the eigenvalues, we substitute the solution into the second boundary condition:

$$v(b) = b^{-1/2} \sin(\sqrt{\lambda - 1/4}\, \ln b) = 0 \Longrightarrow \sqrt{\lambda - 1/4}\, \ln b = n\pi \quad n = 1, 2, 3, \ldots\,,$$

implying that the eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{\ln b}\right)^2 + \frac{1}{4} > \frac{1}{4} \qquad n = 1, 2, \ldots.$$

The eigenfunctions are

$$v_n(x) = x^{-1/2} \sin\left(\frac{n\pi}{\ln b} \ln x\right) \qquad n = 1, 2, 3, \ldots.$$

Since $v_1(x) > 0$ in $(1, b)$ it follows from Proposition 6.41 that λ_1 is indeed the principal eigenvalue.
(b) We apply the method of separation of variables to seek solutions of the form $u = X(x)T(t) \not\equiv 0$. We obtain for X the Sturm–Liouville problem from part (a). For T we obtain

$$T_n(t) = C_n \mathrm{e}^{-\lambda_n t} \qquad n = 1, 2, 3, \ldots$$

where λ_n are given in (a). Therefore, the solution has the form

$$u(x, t) = \sum_{n=1}^{\infty} C_n \mathrm{e}^{-\lambda_n t} x^{-1/2} \sin\left(\frac{n\pi}{\ln b} \ln x\right).$$

The constants C_n are determined by the initial data:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} C_n x^{-1/2} \sin\left(\frac{n\pi}{\ln b} \ln x\right).$$

This is a generalized Fourier series expansion for $f(x)$, and

$$C_n = \frac{\langle f, v_n\rangle}{\langle v_n, v_n\rangle},$$

where $\langle \cdot\,,\cdot\rangle$ denotes the appropriate inner product.

6.5 (a) Notice that under the substitution $y = 1 + x$, $v(y) = u(y-1)$ we obtain

$$(y^2 v')' + \lambda v = 0,$$

where the boundary conditions are

$$v(1) = v(2) = 0 .$$

From here we get (see the solution of Exercise 6.3) that $\lambda > 1/4$,

$$v_n(y) = y^{-1/2} \sin\left(\frac{n\pi \ln y}{\ln 2}\right), \qquad \lambda_n = \frac{n^2\pi^2}{\ln^2 2} + 1/4 \qquad n = 1, 2, \ldots .$$

Therefore,

$$u_n(x) = (x+1)^{-1/2} \sin\left[\frac{n\pi \ln(x+1)}{\ln 2}\right], \qquad \lambda_n = \frac{n^2\pi^2}{\ln^2 2} + 1/4 \qquad n = 1, 2, \ldots$$

(b) Substitute the eigenfunctions that were found in (a) into the inner product

$$\langle u_n, u_k\rangle = \int_1^2 (1+x)^{-1} \sin\left[\frac{n\pi \ln(x+1)}{\ln 2}\right] \sin\left[\frac{k\pi \ln(x+1)}{\ln 2}\right] \mathrm{d}x.$$

Changing variables according to $t = \ln(x+1)$, we find that for $n \neq k$

$$\langle u_n, u_k\rangle = \int_0^{\ln 2} \sin\left(\frac{n\pi t}{\ln 2}\right) \sin\left(\frac{k\pi t}{\ln 2}\right) \mathrm{d}t = 0.$$

6.7 (a) We first verify that all the eigenvalues are positive. For this purpose we multiply the equation by u and integrate by parts using the boundary conditions:

$$0 = \int_1^{\mathrm{e}} u\left[(x^2 u')' + \lambda u\right] \mathrm{d}x = -\int_1^{\mathrm{e}} x^2 (u')^2 \mathrm{d}x + \lambda \int_1^{\mathrm{e}} u^2 \mathrm{d}x.$$

Thus, $u \equiv 0$ if $\lambda < 0$. If $\lambda = 0$, then $u' = 0$ and the boundary conditions imply $u \equiv 0$.

Assume $0 < \lambda < 1/4$. The general solution is

$$u(x) = x^{-1/2}\left(ax^{\sqrt{1-4\lambda}/2} + bx^{-\sqrt{1-4\lambda}/2}\right).$$

The boundary conditions imply again $u = 0$.

Let us check the possibility $\lambda = 1/4$. In this case the general solution is $u(x) = x^{-1/2}(a + b\ln x)$. We can then verify that indeed $1/4$ is not an eigenvalue.

If $\lambda > 1/4$, the general solution is

$$u(x) = x^{-1/2}\left[a\sin\left(\frac{\sqrt{4\lambda-1}}{2}\ln x\right) + b\cos\left(\frac{\sqrt{4\lambda-1}}{2}\ln x\right)\right].$$

Using the boundary conditions we obtain

$$u_n(x) = x^{-1/2}\sin(n\pi\ln x), \qquad \lambda_n = n^2\pi^2 + 1/4 \qquad n = 1, 2, 3, \ldots.$$

Since $u_1(x) > 0$ in $(1, \mathrm{e})$, it follows from Proposition 6.41 that λ_1 is indeed the principal eigenvalue, and therefore there are no eigenvalues λ satisfying $\lambda \le 1/4$.
(b) Substitute the eigenfunctions that were found in (a) into the inner produce

$$\langle u_n, u_k\rangle = \int_1^{\mathrm{e}} x^{-1}\sin(n\pi\ln x)\sin(k\pi\ln x)\,\mathrm{d}x.$$

Changing variables according to $t = \ln x$, we find that for $n \ne k$

$$\langle u_n, u_k\rangle = \int_0^1 \sin n\pi t\,\sin k\pi t\,\mathrm{d}t = 0.$$

6.9 (a) We perform two integration by parts for the expression $\int_{-1}^1 u''v\,\mathrm{d}x$, and use the boundary conditions to handle the boundary terms.
(b) Let u be an eigenfunction associated with the eigenvalue λ. We write the equation that is conjugate to the one satisfied by u:

$$\bar{u}'' + \bar{\lambda}\bar{u} = 0.$$

Obviously $\bar{u}$ satisfies the same boundary conditions as u. Multiply respectively by $\bar{u}$ and by u, and integrate over the interval $[-1, 1]$. Using part (a) we get

$$\lambda\int_{-1}^1 |u(x)|^2\,\mathrm{d}x = \bar{\lambda}\int_{-1}^1 |u(x)|^2\,\mathrm{d}x.$$

Hence λ is real.
(c) Let λ be an eigenvalue. Multiply the ODE by the eigenfunction u, and use the boundary conditions to integrate by parts over $[-1, 1]$. We find

$$\lambda = \frac{\int_{-1}^1 (u')^2\,\mathrm{d}x}{\int_{-1}^1 u^2\,\mathrm{d}x}.$$

Therefore, all the eigenvalues are positive (this can also be checked directly since $\lambda \le 0$ is not an eigenvalue). For $\lambda > 0$ one can readily compute $\lambda_n = \left[(n + \frac{1}{2})\pi\right]^2$ and the eigenfunctions are

$$u_n(x) = a_n \cos\left(n + \frac{1}{2}\right)\pi x + b_n \sin\left(n + \frac{1}{2}\right)\pi x.$$

(d) It follows from part (c) that the multiplicity is 2, and a basis for the eigenspace is

$$\left\{\cos\left(n + \frac{1}{2}\right)\pi x,\ \ \sin\left(n + \frac{1}{2}\right)\pi x\right\}.$$

(e) Indeed the multiplicity is not 1, but this is not a regular Sturm–Liouville problem!

6.11 We represent the solution as $u = v + w$ where w is a particular solution of the inhomogeneous equation

$$\begin{aligned} w_t - w_{xx} + w &= 2t + 15\cos 2x & \qquad 0 < x < \pi/2, \\ w_x(0,t) = w_x(\pi/2,t) &= 0 & t \ge 0. \end{aligned}$$

We write w as $w(x,t) = w_1(x) + w_2(t)$ where

$$-(w_1)'' + w_1 = 15\cos 2x \qquad (w_2)' + w_2 = 2t\ .$$

We obtain

$$w_1(x) = 3\cos 2x \qquad w_2(t) = 2t - 2 + 2\mathrm{e}^{-t}\ .$$

Now, $v = u - w$ solves the homogeneous equation

$$\begin{aligned} v_t - v_{xx} + v &= 0 & \qquad 0 < x < \pi/2, \\ v_x(0,t) = v_x(\pi/2,t) &= 0 & t \ge 0, \\ v(x,0) = u(x,0) - w(x,0) = 1 + \sum_{n=1}^{10} 3n\, \cos 2nx - 3\cos 2x & & 0 \le x \le \pi/2\ . \end{aligned}$$

The solution has the form

$$v(x,t) = \sum_{n=0}^{\infty} B_n \mathrm{e}^{(-4n^2-1)t}\cos 2nx.$$

Substituting $t = 0$ into the proposed solution, we get

$$v(x,0) = \sum_{n=0}^{\infty} B_n \cos 2nx = 1 + \sum_{n=1}^{10} 3n\, \cos 2nx - 3\cos 2x = 1 + \sum_{n=2}^{10} 3n\cos 2nx.$$

Thus,

$$B_0 = 1, \qquad B_n = 3n \quad n = 2,\ldots,10, \qquad B_n = 0 \quad n = 1, 11, 12, \ldots.$$

This implies

$$v(x,t) = \mathrm{e}^{-t} + \sum_{n=2}^{10} 3n\mathrm{e}^{(-4n^2-1)t}\cos 2nx,$$

and the full solution is

$$u(x,t) = \mathrm{e}^{-t} + \sum_{n=2}^{10} 3n\mathrm{e}^{(-4n^2-1)t}\cos 2nx + \left(2t - 2 + 2\mathrm{e}^{-t} + 3\cos 2x\right).$$

The solution is a finite sum of smooth elementary functions, so it is indeed a classical solution.

6.13 To obtain a homogeneous problem, we write

$$u(x,t) = v(x,t) + \frac{xt}{\pi} + 2\left(1 - \frac{x^2}{\pi^2}\right).$$

v solves the system

$$\begin{aligned} v_t - v_{xx} &= xt - 4\pi^{-2} && 0 < x < \pi,\ t > 0,\\ v(0,t) = v(\pi,t) &= 0 && t \geq 0,\\ v(x,0) &= 0 && 0 \leq x \leq \pi. \end{aligned}$$

The solution is

$$v(x,t) = \sum_{n=1}^{\infty} A_n(t)\sin(nx),$$

where $A_n(t)$ satisfies the initial value problem

$$\frac{\mathrm{d}A_n}{\mathrm{d}t} + n^2 A_n = \frac{2}{\pi}\int_0^{\pi}\left(xt - \frac{4}{\pi^2}\right)\sin(nx)\,\mathrm{d}x, \qquad A_n(0) = 0.$$

Computing the integral in the right hand side we obtain

$$\frac{\mathrm{d}A_n}{\mathrm{d}t} + n^2 A_n = \frac{2(-1)^{n+1}}{n}t - \frac{8\left[1-(-1)^n\right]}{n\pi^3}, \qquad A_n(0) = 0.$$

Solving for A_n we get

$$\begin{aligned} A_n(t) &= -\left\{\frac{8\left[1-(-1)^n\right]}{n\pi^3}\right\}\mathrm{e}^{-n^2t}\int_0^t \mathrm{e}^{n^2\tau}\mathrm{d}\tau + \frac{2(-1)^{n+1}}{n}\mathrm{e}^{-n^2t}\int_0^t \tau\mathrm{e}^{n^2\tau}\,\mathrm{d}\tau\\ &= -\left\{\frac{8\left[1-(-1)^n\right]}{n^5\pi^3}\right\}\left(1 - \mathrm{e}^{-n^2t}\right) + \frac{2(-1)^{n+1}}{n^3}\left(t - \frac{1-\mathrm{e}^{-n^2t}}{n^2}\right). \end{aligned}$$

We thus obtain :

$$\begin{aligned} u(x,t) = \sum_{n=1}^{\infty}\Bigg[&-\frac{(2\pi^3+8)(-1)^{n+1}+8}{n^5\pi^3}\left(1 - \mathrm{e}^{-n^2t}\right)\\ &+\frac{2(-1)^{n+1}}{n^3}t\Bigg]\sin(nx) + \frac{xt}{\pi} + 2\left(1 - \frac{x^2}{\pi^2}\right). \end{aligned}$$

6.15 To generate a homogeneous boundary condition we substitute $u(x,t) = v(x,t) + x + t^2$. The initial-boundary value problem for v is

$$\begin{aligned} v_t - v_{xx} &= (9t+31)\sin(3x/2) && 0 < x < \pi, \\ v(0,t) &= v_x(\pi,t) = 0 && t \geq 0, \\ v(x,0) &= 3\pi && 0 \leq x \leq \pi/2. \end{aligned}$$

Its solution is given by

$$v(x,t) = \sum_{n=0}^{\infty} A_n(t)\sin[(n+1/2)x],$$

where

$$\frac{\mathrm{d}A_1}{\mathrm{d}t} + (3/2)^2 A_1 = 9t + 31, \qquad \frac{\mathrm{d}A_n}{\mathrm{d}t} + (n+1/2)^2 A_n = 0 \quad n \neq 1.$$

We find A_i to be

$$\begin{aligned} A_1(t) &= A_1(0)\mathrm{e}^{-9t/4} + 9\mathrm{e}^{-9t/4}\left[\frac{4}{9}\left(t - \frac{4}{9}\right)\mathrm{e}^{9t/4} + \left(\frac{4}{9}\right)^2\right] + \frac{31\times 4}{9}\left(1 - \mathrm{e}^{-9t/4}\right), \\ A_n(t) &= A_n(0)\mathrm{e}^{-(n+1/2)^2 t} \qquad\qquad n \neq 1. \end{aligned}$$

We now use the expansion

$$3\pi = \sum_{n=0}^{\infty} \frac{12}{2n+1}\sin[(n+1/2)x].$$

Comparing coefficients we find

$$A_n(0) = \frac{12}{2n+1}.$$

Thus,

$$v(x,t) = \sum_{n=0}^{\infty} \frac{12}{2n+1}\mathrm{e}^{-(n+1/2)^2 t}\,\sin[(n+1/2)x]$$

$$+\left\{9\mathrm{e}^{-9t/4}\left[\frac{4}{9}\left(t-\frac{4}{9}\right)\mathrm{e}^{9t/4} + \left(\frac{4}{9}\right)^2\right] + \frac{31\times 4}{9}\left(1-\mathrm{e}^{-9t/4}\right)\right\}\sin(3x/2)\ .$$

Finally,

$$u(x,t) = x + t^2 + v(x,t)\ .$$

(b) We obtained a classical solution of the heat equation in the domain $(0,\pi)\times(0,\infty)$. On the other hand, the initial condition does not hold at $x = 0, t = 0$ since it conflicts there with the boundary condition.

6.17 We write $u(x,t) = v(x,t) + x\sin t$. We obtain that v solves

$$\begin{aligned} v_t - v_{xx} &= 1 & 0 < x < 1,\ t > 0, \\ v_x(0,t) = v_x(1,t) &= 0 & t \ge 0, \\ v(x,0) &= 1 + \cos(2\pi x) & 0 \le x \le 1. \end{aligned}$$

The solution's structure is

$$v(x,t) = A_0(t) + \sum_{n=1}^{\infty} A_n(t)\cos n\pi x,$$

where

$$\frac{\mathrm{d}A_0}{\mathrm{d}t} = 1, \quad \frac{\mathrm{d}A_n}{\mathrm{d}t} + (n\pi)^2 A_n = 0 \qquad n \ge 1,$$
$$A_0(0) = A_2(0) = 1\ , \quad A_n(0) = 0 \qquad \forall n \ne 0, 2.$$

We obtain at once

$$A_0(t) = 1 + t \qquad A_2(t) = \mathrm{e}^{-4\pi^2 t}, \qquad A_n(t) = 0 \quad \forall n \ne 0, 2.$$

Thus,

$$u(x,t) = x\sin(t) + 1 + t + \mathrm{e}^{-4\pi^2 t}\cos(2\pi x).$$

(b) The solution is classic in the domain $[0,1] \times [0,\infty)$.

6.18 The solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n(t)\sin\left(\frac{n\pi x}{2}\right),$$

where

$$\frac{\mathrm{d}^2 A_n}{\mathrm{d}t^2} + \frac{\mathrm{d}A_n}{\mathrm{d}t} + \left(\frac{n\pi}{2}\right)^2 A_n = 0,$$
$$A_n(0) = 0\ , \quad \frac{\mathrm{d}A_n(0)}{\mathrm{d}t} = \int_0^2 x\sin\left(\frac{n\pi x}{2}\right)\mathrm{d}x = \frac{4(-1)^{n+1}}{n\pi}.$$

We obtain the solution

$$A_n(t) = \frac{8(-1)^{n+1}\mathrm{e}^{-t/2}}{n\pi\sqrt{(n\pi)^2 - 1}}\sin\left(\frac{\sqrt{(n\pi)^2 - 1}\,t}{2}\right),$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8(-1)^{n+1}\mathrm{e}^{-t/2}}{n\pi\sqrt{(n\pi)^2 - 1}}\sin\left(\frac{\sqrt{(n\pi)^2 - 1}\,t}{2}\right)\sin\left(\frac{n\pi x}{2}\right).$$

(b) No. The boundary condition $u(2,t) = 0$ is not compatible with the initial condition $u_t(x,0) = x$ at the point $x = 2$, $t = 0$.

6.19 To obtain a homogeneous boundary condition we write $v(x,t) = a(t)x + b(t)$. We find $v(x) = x/\pi$. Define now $w(x,t) = u(x,t) - v(x)$ and formulate an initial-boundary value problem for w:

$$\begin{aligned} w_t - w_{xx} + hw &= -\frac{hx}{\pi} && 0 < x < \pi,\ t > 0, \\ w(0,t) = w(\pi,t) &= 0 && t \geq 0, \\ w(x,0) = u(x,0) - v(x) &= -\frac{x}{\pi} && 0 \leq x \leq \pi. \end{aligned}$$

We write the expansion for w as

$$w(x,t) = \sum_{n=0}^{\infty} T_n(t)\widetilde{X}_n(x)\,,$$

where $\widetilde{X}_n$ are the eigenfunctions of the associated Sturm–Liouville problem, namely

$$\lambda_n = n^2\,, \qquad \widetilde{X}_n(x) = \sin nx \quad n = 1,2,3,\ldots$$

Using the expansion of w in terms of $\widetilde{X}_n$ we obtain

$$\sum_{n=1}^{\infty} \left[T_n(t)' + (n^2+h)T_n(t)\right] \sin nx = -\frac{hx}{\pi}\,.$$

We proceed to expand $f(x) = x$ into a sine series in the interval $[0,\pi]$

$$x = \sum_{n=1}^{\infty} B_n \sin nx\,,$$

$$B_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx \,\mathrm{d}x = \frac{2}{\pi}\left(\frac{-nx\cos nx + \sin nx}{n^2}\right)\Bigg|_0^{\pi} = 2\frac{(-1)^{n+1}}{n}\,.$$

Substituting this expansion into the PDE, we obtain a sequence of ODEs:

$$T_n(t)' + (n^2+h)T_n(t) = \frac{2(-1)^n h}{n\pi} \qquad n = 1,2,3,\ldots$$

whose solutions are

$$T_n(t) = A_n \mathrm{e}^{-(n^2+h)t} + \frac{2(-1)^n h}{n\pi(n^2+h)}\,.$$

The constants A_n will be determined later on. Therefore,

$$w(x,t) = \sum_{n=1}^{\infty}\left(A_n \mathrm{e}^{-(n^2+h)t} + \frac{2(-1)^n h}{n\pi(n^2+h)}\right)\sin nx\,.$$

We proceed to find A_n from the initial condition

$$w(x,0) = \sum_{n=1}^{\infty}\left[A_n + \frac{2(-1)^n h}{n\pi(n^2+h)}\right]\sin nx = -\frac{x}{\pi} = \sum_{n=1}^{\infty}\frac{2(-1)^n}{n\pi}\,\sin nx\,.$$

Therefore,

$$A_n = \frac{2(-1)^n}{n\pi}\left(1 - \frac{h}{n^2+h}\right) \qquad n = 1, 2, 3, \ldots .$$

It follows that

$$w(x,t) = \sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi}\left[\left(1 - \frac{h}{n^2+h}\right)\mathrm{e}^{-(n^2+h)t} + \frac{h}{n^2+h}\right]\sin nx \,,$$

and $u(x,t) = w(x,t) + v(x,t) = w(x,t) + \frac{x}{\pi}$.

This solution is not classical at $t = 0$, since the sine series does not converge to $-x/\pi$ in the closed interval $[0,1]$.

6.21 We seek a particular solution to the PDE of the form $v(x,t) = f(t)\cos(2001x)$. The equation implies

$$v_t - v_{xx} = f'(t)\cos 2001x + 2001^2 f(t)\cos 2001x = t\cos 2001x \,.$$

Therefore, $f(t)$ solves the ODE

$$f(t)' + 2001^2 f(t) = t \,,$$

and we obtain

$$f(t) = \frac{t}{2001^2} - \frac{1}{2001^4} \,,$$

$\Rightarrow$

$$v(x,t) = \left(\frac{t}{2001^2} - \frac{1}{2001^4}\right)\cos 2001x \,.$$

Set $w(x,t) = u(x,t) - v(x,t)$, and write for w:

$$\begin{aligned} w_t - w_{xx} &= 0 && 0 < x < \pi,\ t > 0, \\ w_x(0,t) = w_x(\pi,t) &= 0 && t \geq 0, \\ w(x,0) = u(x,0) - v(x,0) &= \pi\cos 2x + \frac{\cos 2001x}{2001^4} && 0 \leq x \leq \pi. \end{aligned}$$

Expand w into an eigenfunctions series

$$w(x,t) = \sum_{n=0}^{\infty} T_n(t)\cos nx \,,$$

where $T_n(t)$ solves

$$T_n(t)' + n^2 T_n(t) = 0 \qquad n = 0, 1, 2, \ldots .$$

We find

$$T_0(t) = A_0, \quad T_n(t) = A_n\mathrm{e}^{-n^2 t} \quad n = 1, 2, 3, \ldots,$$

implying

$$w(x,t) = A_0 + \sum_{n=1}^{\infty} A_n\mathrm{e}^{-n^2 t}\cos nx \,.$$

Evaluating the sum at $t = 0$

$$w(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos nx = \pi \cos 2x + \frac{1}{2001^4} \cos 2001x \,,$$

and comparing coefficients we get

$$A_2 = \pi, \quad A_{2001} = \frac{1}{2001^4}, \qquad A_n = 0 \quad n \neq 2, 2001.$$

Finally we write

$$u(x,t) = \pi e^{-4t} \cos 2x + \frac{1}{2001^4} e^{-2001^2 t} \cos 2001x + \left(\frac{t}{2001^2} - \frac{1}{2001^4} \right) \cos 2001x.$$

6.22 Write $v(x,t) = a(t)x^2 + b(t)x + c(t)$ to obtain from the boundary conditions the function $v(x,t) = x^2/2 + c(t)$. If we demand v to solve the homogeneous PDE too, we further find

$$v_t - 13v_{xx} = c'(t) - 13 = 0, \implies c(t) = 13t.$$

Set $w(x,t) = u(x,y) - v(x,t)$ and substitute into the initial-boundary value problem:

$$\begin{aligned} w_t - 13w_{xx} &= 0 & 0 < x < 1,\ t > 0, \\ w_x(0,t) = w_x(1,t) &= 0 & t \geq 0, \\ w(x,0) = u(x,0) - v(x,0) &= x & 0 \leq x \leq 1. \end{aligned}$$

The relevant eigenfunctions are $X_n = \cos n\pi x$, implying

$$w(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-13n^2\pi^2 t} \cos n\pi x.$$

The initial conditions then lead to $w(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos n\pi x = x$. Thus,

$$A_0 = \int_0^1 x \, dx = \left. \frac{x^2}{2} \right|_0^1 = \frac{1}{2},$$

$$A_n = 2 \int_0^1 x \cos n\pi x \, dx = \frac{2}{n^2\pi^2} \left[(-1)^n - 1 \right],$$

and the solution is

$$u(x,t) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{e^{-13(2k-1)^2\pi^2 t}}{(2k-1)^2} \cos (2k-1)\pi x + \frac{x^2}{2} + 13t \,.$$

6.23 (a) A particular solution to the PDE is given by

$$v(x,t) = Ae^{3t} \cos 17\pi x,$$

where A satisfies

$$3Ae^{3t}\,\cos 17\pi x + 17^2\pi^2 Ae^{3t}\,\cos 17\pi x = e^{3t}\,\cos 17\pi x\;.$$

Therefore, $A = 1/(3+17^2\pi^2)$. Note that v satisfies the boundary conditions. We set $w(x,t) = u(x,t) - v(x,t)$ and obtain for w

$$\begin{aligned} w_t - w_{xx} &= 0 && 0<x<1\,, t>0\,,\\ w_x(0,t) = w_x(1,t) &= 0 && t\geq 0,\\ w(x,0) = 3\cos 42\pi x - \frac{1}{3+17^2\pi^2}\,\cos 17\pi x & && 0\leq x\leq 1. \end{aligned}$$

Solving for w:

$$w(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 t}\cos n\pi x\;,$$

where $\{A_n\}$ are found from the initial conditions

$$w(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos n\pi x = 3\cos 42\pi x - \frac{1}{3+17^2\pi^2}\,\cos 17\pi x\;.$$

We conclude

$$A_{17} = -\frac{1}{3+17^2\pi^2}\,, \qquad A_{42} = 3\,, \qquad A_n = 0 \qquad \forall\, n\neq 17, 42.$$

Therefore,

$$u(x,t) = -\frac{e^{-17^2\pi^2 t}\cos 17\pi x}{3+17^2\pi^2} + 3e^{-42^2\pi^2 t}\cos 42\pi x + \frac{e^{3t}\cos 17\pi x}{3+17^2\pi^2}.$$

(b) The general solution takes the form

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 t}\cos n\pi x\;.$$

The function $f(x) = 1/(1+x^2)$ is continuous in $[0,1]$, implying that A_n are all bounded. Therefore, the series converges uniformly for all $t > t_0 > 0$. Hence,

$$\lim_{t\to\infty} u(x,t) = A_0 = \int_0^{\pi} \frac{dx}{1+x^2} = \frac{\pi}{4}\;.$$

6.24 Substituting the expansion

$$u(x,t) = \sum_{n=0}^{\infty} T_n(t)\cos nx$$

into the PDE we obtain

$$\sum_{n=0}^{\infty}\left[(T_n)''(t)\cos nx + n^2 T_n(t)\cos nx\right] = \cos 2t\,\cos 3x\,,$$

leading to

$$\begin{aligned}(T_n)''(t) + n^2\,T_n(t) &= 0 \qquad && n \neq 3\,,\\ (T_3)''(t) + 9\,T_3(t) &= \cos 2t \qquad && n = 3\,.\end{aligned}$$

Solving the ODEs we find

$$\begin{aligned}T_0(t) &= A_0 t + B_0,\\ T_3(t) &= A_3\cos 3t + B_3\sin 3t + \frac{1}{5}\cos 2t,\\ T_n(t) &= A_n\cos nt + B_n\sin nt \qquad n \neq 0,3.\end{aligned} \tag{12.25}$$

Therefore,

$$u(x,t) = \frac{1}{5}\cos 2t\,\cos 3x + (A_0 t + B_0) + \sum_{n=1}^{\infty}(A_n\cos nt + B_n\sin nt)\cos nx\,.$$

The first initial condition

$$u(x,0) = \frac{1}{5}\cos 3x + B_0 + \sum_{n=1}^{\infty}A_n\cos nx = \cos^2 x = \frac{1}{2}(\cos 2x + 1)$$

implies

$$A_3 = -\frac{1}{5}\,,\quad A_2 = \frac{1}{2}\,,\quad B_0 = \frac{1}{2}\,,\quad A_n = 0 \qquad \forall n \neq 0,2,3.$$

The second initial condition

$$u_t(x,0) = A_0 + \sum_{n=1}^{\infty} nB_n\cos nx = 1$$

implies $A_0 = 1$, and $B_n = 0$ for all $n \neq 0$. Therefore,

$$u(x,t) = \frac{1}{5}\cos 2t\,\cos 3x + t + \frac{1}{2} + \frac{1}{2}\cos 2t\,\cos 2x - \frac{1}{5}\cos 3t\,\cos 3x\,.$$

6.25 Seeking a particular solution $v(t)$ that satisfies also the boundary condition we write

$$v_t(t) = \alpha\cos\omega t\,, \Longrightarrow v(t) = \frac{\alpha}{\omega}\sin\omega t.$$

We set $w(x,t) = u(x,t) - v(t)$ and formulate a new problem for w:

$$\begin{aligned}w_t - kw_{xx} &= 0 \qquad && 0 < x < L\,, t > 0\,,\\ w_x(0,t) = w_x(L,t) &= 0 && t \geq 0\,,\\ w(x,0) = u(x,0) - v(0) &= x && 0 \leq x \leq L\,.\end{aligned}$$

The solution takes the form

$$w(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \mathrm{e}^{-k\frac{n^2\pi^2}{L^2}t} \cos\frac{n\pi x}{L} \,.$$

The coefficients A_n are determined by the initial conditions

$$w(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} = x \,,$$

$$\begin{aligned} A_0 &= \frac{1}{L}\int_0^L x\,\mathrm{d}x = \frac{x^2}{2L}\bigg|_0^L = \frac{L}{2}, \\ A_n &= \frac{2}{L}\int_0^L x\cos\frac{n\pi x}{L}\,\mathrm{d}x = \frac{2L}{n^2\pi^2}\left[(-1)^n - 1\right]. \end{aligned}$$

Therefore,

$$u(x,t) = w(x,t) + v(t) = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{m=1}^{\infty}\frac{\mathrm{e}^{-k\frac{(2m-1)^2\pi^2}{L^2}t}}{(2m-1)^2}\cos\frac{(2m-1)\pi x}{L} + \frac{\alpha}{\omega}\sin\omega t \,.$$

6.26 The function $v(x) = (2\pi - 1)x + 1$ satisfies the given boundary conditions. We thus define $w(x,t) = u(x,t) - v(x)$ and formulate for w the new problem

$$\begin{aligned} w_{tt} - c^2 w_{xx} &= 0 && 0 < x < 1\,, t > 0\,, \\ w(0,t) = w(1,t) &= 0 && t \geq 0\,, \\ w(x,0) = u(x,0) - v(x) &= 2(1-\pi)(x - 1/2) && 0 \leq x \leq 1\,, \\ w_t(x,0) = u_t(x,0) &= 0 && 0 \leq x \leq 1\,. \end{aligned}$$

The solution is

$$w(x,t) = \sum_{n=1}^{\infty}(A_n\cos cn\pi t + B_n \sin cn\pi t)\sin n\pi x \,.$$

We use the initial conditions to determine A_n and B_n:

$$\begin{aligned} w(x,0) &= \sum_{n=1}^{\infty} A_n \sin n\pi x = 2(1-\pi)(x - 1/2)\,, \\ w_t(x,0) &= \sum_{n=1}^{\infty} B_n cn\pi \sin n\pi x = 0\,. \end{aligned}$$

We conclude that $B_n = 0$ for all n, and

$$\frac{A_n}{2(1-\pi)} = 2\int_0^1 (x - \frac{1}{2})\sin n\pi x\,\mathrm{d}x = -\frac{1}{n\pi}\left[(-1)^n + 1\right] \,.$$

Therefore, the solution is

$$u(x,t) = -\sum_{k=1}^{\infty} \frac{2(1-\pi)}{k\pi} \cos\left(2ck\pi t\right) \sin\left(2k\pi x\right) \; + \; (2\pi - 1)x + 1\,.$$

The solution is not classical. This can be seen either by observing that the initial conditions are not compatible with the boundary conditions, or by checking that the differentiated series does not converge at every point.

6.27 The PDE is equivalent to

$$ru_t = ru_{rr} + 2u_r.$$

We set

$$w(r,t) := u(r,t) - a,$$

and obtain for w

$$\begin{array}{ll} rw_t = rw_{rr} + 2w_r & 0 < r < a,\ t > 0, \\ w(a,t) = 0 & t \geq 0, \\ w(r,0) = r - a & 0 \leq r \leq (a) \end{array} \tag{12.26}$$

We solve for w by the method of separation of variables: $w(r,t) = R(r)T(t)$. We find for R

$$rR'' + 2R' + \lambda rR = 0.$$

It is convenient to define $\rho(r) = rR(r)$. This implies $\rho(0) = 0$ and

$$\begin{cases} \rho'' + \lambda\,\rho = 0 & 0 < r < a, \\ \rho(0) = \rho(a) = 0, \end{cases} \tag{12.27}$$

The eigenvalues and eigenfunctions of (12.27) are $\lambda_n = n^2\pi^2/a^2$, $\rho_n(r) = \sin(n\pi r/a)$, where $n \geq 1$. Therefore,

$$R_n(r) = \frac{1}{r}\sin\frac{n\pi r}{a}.$$

Substituting λ_n into the equation for T we derive $T_n(t) = \exp(-n^2\pi^2 t/a^2)$, and the solution takes the form

$$w(r,t) = \sum_{n=1}^{\infty} A_n\, \mathrm{e}^{-\frac{n^2\pi^2 t}{a^2}}\, \frac{1}{r}\sin\frac{n\pi r}{a}. \tag{12.28}$$

The initial conditions then imply

$$w(r,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi r}{a} = r\ (r-a).$$

Therefore, A_n are the (generalized) Fourier coefficients of $r(r-a)$, i.e.

$$A_n = \frac{2}{a}\int_0^a r\ (r-a)\sin\frac{n\pi r}{a}\ \mathrm{d}r = -\frac{4\ a^2}{n^3\ \pi^3}[1-(-1)^n].$$

Chapter 7

7.1 Select $\vec{\psi} = v\vec{\nabla} u$ in Gauss' theorem:

$$\int_D \vec{\nabla} \cdot \vec{\psi}(x,y)\,\mathrm{d}x\mathrm{d}y = \int_{\partial D} \vec{\psi}(x(s),y(s)) \cdot \hat{n}\mathrm{d}s.$$

7.3 We solve by the separation of variables method: $u(x,y) = X(x)Y(y)$. We obtain

$$X''Y + Y''X = kXY \quad \Rightarrow \quad \frac{-Y''}{Y} = \frac{X''}{X} - k = \lambda.$$

We derive for Y a Sturm–Liouville problem

$$Y'' + \lambda Y = 0, \;\; Y(0) = Y(\pi) = 0.$$

Therefore, the eigenvalues and eigenfunctions are

$$\lambda_n = n^2, \;\; Y_n(y) = \sin ny \qquad n = 1, 2 \ldots.$$

Then, for X we obtain

$$(X_n)'' - (k + n^2)X_n = 0 \quad \Rightarrow \quad X_n(x) = A_n \mathrm{e}^{\sqrt{(k+n^2)}\,x} + B_n \mathrm{e}^{-\sqrt{(k+n^2)}\,x}.$$

The general solution is thus

$$u(x,y) = \sum_{n=1}^{\infty} \left[A_n \mathrm{e}^{\sqrt{(k+n^2)}\,x} + B_n \mathrm{e}^{-\sqrt{(k+n^2)}\,x} \right] \sin ny.$$

The boundary conditions in the x direction are expressed as

$$u(0,y) = \sum_{n=1}^{\infty} (A_n + B_n) \sin ny = 1, \tag{12.29}$$

$$u(\pi,y) = \sum_{n=1}^{\infty} \left[A_n \mathrm{e}^{\sqrt{(k+n^2)}\,\pi} + B_n \mathrm{e}^{-\sqrt{(k+n^2)}\,\pi} \right] \sin ny = 0. \tag{12.30}$$

We expand $f(y) = 1$ into a sine series

$$1 = \sum_{n=1}^{\infty} b_n \sin ny, \quad b_n = \frac{2}{\pi} \int_0^{\pi} \sin(ny)\,\mathrm{d}y = \frac{-2}{\pi n}[(-1)^n - 1]. \tag{12.31}$$

Comparing coefficients yields

$$A_n = -\frac{b_n \,\mathrm{e}^{-\sqrt{(k+n^2)}\,\pi}}{\mathrm{e}^{\sqrt{(k+n^2)}\,\pi} - \mathrm{e}^{-\sqrt{(k+n^2)}\,\pi}}, \qquad B_n = \frac{b_n \,\mathrm{e}^{\sqrt{(k+n^2)}\,\pi}}{\mathrm{e}^{\sqrt{(k+n^2)}\,\pi} - \mathrm{e}^{-\sqrt{(k+n^2)}\,\pi}}.$$

Together with (12.31) we finally write

$$u(x,y) = \frac{4}{\pi}\sum_{l=1}^{\infty} \frac{\sinh\left[\sqrt{k+(2l-1)^2}\,(\pi - x)\right]}{(2l-1)\sinh\left[\sqrt{k+(2l-1)^2}\,\pi\right]}\sin\left[(2l-1)y\right].$$

7.5 We should show that

$$M(r_1) < M(r_2) \qquad \forall\ 0 < r_1 < r_2 < R.$$

Let $B_r = \{(x,y) \mid x^2+y^2 \le r^2\}$ be a disk of radius r. Choose arbitrary $0 < r_1 < r_2 < R$. Since $u(x,y)$ is a nonconstant harmonic function in B_R, it must be a nonconstant harmonic function in each sub-disk. The strong maximum principle implies that the maximal value of u in the disk B_{r_2} is obtained only on the disk's boundary. As all the points in B_{r_1} are internal to B_{r_2}, we have

$$u(x,y) < \max_{(x,y)\in\partial B_{r_2}} u(x,y) = M(r_2) \qquad \forall\ (x,y) \in B_{r_1}.$$

In particular,

$$M(r_1) = \max_{(x,y)\in\partial B_{r_1}} u(x,y) < M(r_2).$$

7.7 (a) The Laplace equation in cartesian coordinates is

$$\Delta w = w_{xx} + w_{yy} = 0.$$

We change variables into

$$x = r\cos\theta, \quad y = r\sin\theta, \quad u(r,\theta) := w(x(r,\theta), y(r,\theta)).$$

The inverse transformation is given by

$$\begin{cases} r = \sqrt{x^2+y^2}\,, \\ \theta = \arctan\left(y/x\right). \end{cases} \tag{12.32}$$

By the chain rule we obtain

$$\begin{aligned} w_{xx} &= u_{rr}r_x^2 + 2u_{r\theta}r_x\theta_x + u_{\theta\theta}\theta_x^2 + u_r r_{xx} + u_\theta\theta_{xx}, \\ w_{yy} &= u_{rr}r_y^2 + 2u_{r\theta}r_y\theta_y + u_{\theta\theta}\theta_y^2 + u_r r_{yy} + u_\theta\theta_{yy}. \end{aligned}$$

From (12.32):

$$\begin{aligned} &r_x = \tfrac{x}{\sqrt{x^2+y^2}}\,, \qquad r_y = \tfrac{y}{\sqrt{x^2+y^2}}\,, \qquad \theta_x = \tfrac{-y}{x^2+y^2}\,, \qquad \theta_y = \tfrac{x}{x^2+y^2}\,, \\ &r_{xx} = \tfrac{y^2}{(x^2+y^2)^{3/2}}\,, \quad r_{yy} = \tfrac{x^2}{(x^2+y^2)^{3/2}}\,, \quad \theta_{xx} = \tfrac{2xy}{(x^2+y^2)^2}\,, \quad r_{yy} = \tfrac{-2xy}{(x^2+y^2)^2}\,. \end{aligned}$$

Therefore,

$$w_{xx}+w_{yy}=u_{rr}(r_x^2+r_y^2)+2u_{r\theta}(r_x\theta_x+r_y\theta_y)+u_{\theta\theta}(\theta_x^2+\theta_y^2)$$
$$+u_r(r_{xx}+r_{yy})+u_\theta(\theta_{xx}+\theta_{yy})=u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}.$$

(b) In polar coordinates

$$x=r\cos\theta,\quad y=r\sin\theta,\quad 0<r<\sqrt{6},\quad -\pi\le\theta<\pi$$

we obtain the problem

$$\begin{aligned} u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}&=0 && 0<r<\sqrt{6},\quad -\pi\le\theta<\pi,\\ u(\sqrt{6},\theta)&=\sqrt{6}\,\sin\theta+6\,\sin^2\theta && -\pi\le\theta\le\pi. \end{aligned}$$

The general solution takes the form

$$u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n(a_n\cos n\theta+b_n\sin n\theta),\quad R=\sqrt{6}.$$

The boundary condition implies

$$u(\sqrt{6},\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)=3+\sqrt{6}\,\sin\theta-3\cos 2\theta.$$

Equating coefficients leads to

$$a_0=6,\quad a_2=-3,\quad a_n=0\quad \forall n\neq 0,2,\ \text{and}\ \ b_1=\sqrt{6},\quad b_n=0\quad \forall n\neq 1.$$

Therefore, the solution is

$$u(r,\theta)=3-\frac{r^2}{2}\cos 2\theta+r\,\sin\theta=3-r^2\cos^2\theta+\frac{r^2}{2}+r\,\sin\theta,$$

or, in cartesian coordinates,

$$u(x,y)=3+y+\frac{1}{2}(y^2-x^2).$$

7.9 $n=0$: A homogeneous harmonic polynomial is of the form $P_0(x,y)=c$ and the dimension of V_0 is 1.
$n\ge 1$: A homogeneous harmonic polynomial has the following form in polar coordinates:

$$u(r,\theta)=P_n(r,\theta)=\sum_{i+j=n}a_{i,j}(r\cos\theta)^i(r\sin\theta)^j,$$

Hence,

$$u(r,\theta)=r^n\sum_{i+j=n}a_{i,j}(\cos\theta)^i(\sin\theta)^j=r^nf(\theta).$$

Substitute $u(r,\theta)$ into the Laplace equation to obtain

$$f(\theta) = f_n(\theta) = A_n \cos n\theta + B_n \sin n\theta,$$

implying that

$$P_n(r,\theta) = r^n f(\theta) = r^n (A_n \cos n\theta + B_n \sin n\theta).$$

It follows that the homogeneous harmonic polynomials of order $n \geq 1$ are spanned by two basis functions:

$$v_1(r,\theta) = r^n \cos n\theta \;; \qquad v_2(r,\theta) = r^n \sin n\theta,$$

and the dimension of V_n (for $n \geq 1$) is 2.

7.11 The general harmonic function has the form

$$u(r,\theta) = (C_0 \,\ln r + D_0) + \sum_{n=1}^{\infty} (C_n r^n + D_n r^{-n})\,(A_n \cos n\theta + B_n \sin n\theta).$$

Since we seek bounded solutions we require $C_n = 0$ for $n \geq 0$, and obtain

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{R}{r}\right)^n (a_n \cos n\theta + b_n \sin n\theta). \tag{12.33}$$

Using the boundary condition we get

$$u(r,\theta) = \frac{2}{r}\, 2\, \sin\theta = \frac{4}{r^2}\, r\, \sin\theta,$$

or, in cartesian coordinates,

$$u(x,y) = \frac{4y}{x^2+y^2}.$$

7.13 Consider the function

$$g(\varphi) = \frac{a^2 - r^2}{a^2 - 2\,a\,r\,\cos(\theta-\varphi) + r^2}$$

in the interval $[-\pi,\ \pi]$. It is easy to check that

$$-1 \leq \cos(\theta-\varphi) \leq 1 \;\;\Rightarrow\;\; 2ar \geq -2ar\cos(\theta-\varphi) \geq -2ar,$$

and thus

$$a^2 + 2\,a\,r + r^2 \geq a^2 - 2\,a\,r\,\cos(\theta-\varphi) + r^2 \geq a^2 - 2\,a\,r + r^2.$$

Therefore, we obtain for $0 < r < a$ that

$$\frac{a^2-r^2}{a^2+2\,a\,r+r^2} \leq\; g(\varphi)\; \leq \frac{a^2-r^2}{a^2-2\,a\,r+r^2},$$

or

$$\frac{a-r}{a+r} \le\; g(\varphi)\; \le \frac{a+r}{a-r}\,. \tag{12.34}$$

The Poisson integral representation for $f \ge 0$, and (12.34) imply

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\,\frac{a-r}{a+r}\,\mathrm{d}\varphi \;\le\; u(r,\theta) \;\le\; \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\,\frac{a+r}{a-r}\,\mathrm{d}\varphi,$$

and thus

$$\left(\frac{a-r}{a+r}\right)\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\,\mathrm{d}\varphi \;\le\; u(r,\theta) \;\le\; \left(\frac{a+r}{a-r}\right)\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\,\mathrm{d}\varphi.$$

By the mean value theorem

$$\frac{a-r}{a+r}u(0,0) \le u(r,\theta) \le \frac{a+r}{a-r}u(0,0).$$

7.15 (a) Suppose v has a local maximum at $(x_0,y_0) \in D$. Then

$$v_x(x_0,y_0) = v_y(x_0,y_0) = 0, \qquad v_{xx}(x_0,y_0) \le 0\,, v_{yy}(x_0,y_0) \le 0.$$

Therefore, at this point $v_{xx}+v_{yy}+xv_x+yv_y \le 0$, which is a contradiction.
(b) Let $\varepsilon > 0$. The function v_ε satisfies

$$(v_\varepsilon)_{xx} + (v_\varepsilon)_{yy} + x(v_\varepsilon)_x + y(v_\varepsilon)_y > 0\,,$$

and thus according to part (a) the maximum of v_ε is obtained on ∂D. Let M be the maximum of u on ∂D. For all $(x_1,y_1) \in D$

$$u(x_1,y_1) \le v_\varepsilon(x_1,y_1) \le \max\{v_\varepsilon(x,y)\,|\,(x,y) \in \partial D\} \le M + \varepsilon\pi^2.$$

Letting $\varepsilon \to 0$, we obtain $u(x_1,y_1) \le M$.
(c) Write $w(x,y) := u_1(x,y) - u_2(x,y)$, where $u_1(x,y), u_2(x,y)$ are two solutions of the problem. We should show that $w(x,y) = 0$ in D. Notice that the functions $\pm w(x,y)$ solve the equation with homogeneous boundary conditions. Therefore, part (b) implies $\pm w(x,y) \le 0$ in D, namely $w(x,y) = 0$ in D.

7.17 (a) The general solution is of the form

$$u(x,t) = \sum_{n=1}^{\infty} B_n \mathrm{e}^{-2n^2 t}\sin nx. \tag{12.35}$$

Substituting the initial condition into (12.35) we write

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin nx = x(x^2-\pi^2). \tag{12.36}$$

To find B_n we expand $u(x,0) = f(x) = x(x^2 - \pi^2)$ into an eigenfunction series:

$$B_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx \,\mathrm{d}x = \frac{12(-1)^n}{n^3}.$$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \mathrm{e}^{-2n^2 t}\sin nx. \tag{12.37}$$

(b) Since f and f' are continuous and furthermore $f(0) = f(\pi) = 0$, the series (12.36) converges uniformly to the function f. By Corollary 7.18, u solves the heat equation in D.

7.19 (a) The mean value theorem for harmonic functions implies

$$u(0,0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(r,\theta)\,\mathrm{d}\theta$$

for all $0 < r \leq R$. Substitute $r = R$ into the equation above to obtain

$$u(0,0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(R,\theta)\,\mathrm{d}\theta = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \sin^2(2\theta)\,\mathrm{d}\theta = \frac{1}{4}\ .$$

(b) This is an immediate consequence of the strong maximum principle. This principle implies

$$u(r,\theta) \leq \max_{\psi\in[-\pi/2,\pi/2)} u(R,\psi) = 1$$

for all $r < R$, and the equality holds if and only if u is constant. Clearly our solution is not a constant function, and therefore $u < 1$ in D. The inequality $u > 0$ is obtained from the strong maximum principle applied to $-u$.

7.21 The function $w(x,t) = \mathrm{e}^{-t}\sin x$ solves the problem

$$\begin{aligned} w_t - w_{xx} &= 0 \qquad && (x,t)\in Q_T,\\ w(0,t) = w(\pi,t) &= 0 && 0\leq t\leq T\,,\\ w(x,0) &= \sin(x) && 0\leq x\leq \pi\,. \end{aligned}$$

On the parabolic boundary $0 \leq u(x,t) \leq w(x,t)$, and therefore, from the maximum principle $0 \leq u(x,t) \leq w(x,t)$ in the entire rectangle Q_T.

Chapter 8

8.1 (a) Fix $(\xi,\eta)\in B_R$. Recall that for $(x,y)\in B_R\setminus(\xi,\eta)$ we have

$$G_R(x,y;\xi,\eta)=\begin{cases}-\frac{1}{2\pi}\ln\frac{Rr}{\rho r^*} & (\xi,\eta)\neq(0,0),\\ -\frac{1}{2\pi}\ln\frac{r}{R} & (\xi,\eta)=(0,0),\end{cases} \tag{12.38}$$

where

$$r=\sqrt{(x-\xi)^2+(y-\eta)^2},\ r^*=\sqrt{(x-\frac{R^2}{\rho^2}\xi)^2+(y-\frac{R^2}{\rho^2}\eta)^2},\ \rho=\sqrt{\xi^2+\eta^2}.$$

Assume first that $(\xi,\eta)=(0,0)$. It is easy to check that

$$G_R(x,y;0,0)\mid_{x^2+y^2=R^2}=0.$$

On the other hand, $G_R(x,y;0,0)=\Gamma(x,y)+\text{constant}$, therefore,

$$\Delta\left(\frac{-1}{2\pi}\ln\frac{r}{R}\right)=-\delta(x,y).$$

Suppose now that $(\xi,\eta)\neq(0,0)$. Then

$$G_R(x,y;\xi,\eta)=\Gamma(x-\xi,y-\eta)-\Gamma(R^{-1}\sqrt{\xi^2+\eta^2}(x-\tilde{\xi},y-\tilde{\eta})).$$

Since $(\tilde{\xi},\tilde{\eta})\notin B_R$, it follows that $\Gamma(R^{-1}\sqrt{\xi^2+\eta^2}(x-\tilde{\xi},y-\tilde{\eta}))$ is harmonic in B_R. On the other hand, for $(x,y)\in\partial B_R$ we have $\Gamma(R^{-1}\sqrt{\xi^2+\eta^2}(x-\tilde{\xi},y-\tilde{\eta}))=\Gamma(x-\xi,y-\eta)$. Therefore, $G_R(x,y;\xi,\eta)$ is the Green function in B_R.

Now, using polar coordinates (r,θ) for (x,y), and (R,ϕ) for (ξ,η), we obtain

$$\frac{\partial G_R(x,y;\xi,\eta)}{\partial\xi}=\frac{\xi(1-r^2/R^2)}{2\pi(R^2-2Rr\cos(\theta-\phi)+r^2)},$$

and similarly for $\partial/\partial\eta$. The exterior unit normal at a point (ξ,η) on the sphere is $(\xi,\eta)/R$, therefore,

$$\frac{\partial G_R(x,y;\xi,\eta)}{\partial r}=\frac{R^2-r^2}{2\pi R(R^2-2Rr\cos(\theta-\phi)+r^2)}.$$

(b) Using (12.38) it follows that $\lim_{R\to\infty}G_R(x,y;\xi,\eta)=\infty$.

8.2 Fix two points $(x,y),(\xi,\eta)\in D$ such that $(x,y)\neq(\xi,\eta)$, and let

$$v(\sigma,\tau):=N(\sigma,\tau;x,y),\qquad w(\sigma,\tau):=N(\sigma,\tau;\xi,\eta).$$

The functions v and w are harmonic in $D\setminus\{(x,y),\ (\xi,\eta)\}$ and satisfy

$$\partial_n v(\sigma,\tau)=\partial_n w(\sigma,\tau)=-\frac{1}{L}\qquad(\sigma,\tau)\in\partial D,$$

and

$$\int_{\partial D} v(\sigma,\tau)\,\mathrm{d}s(\sigma,\tau) = \int_{\partial D} w(\sigma,\tau)\,\mathrm{d}s(\sigma,\tau) = 0.$$

Therefore,

$$\int_{\partial D} (w\partial_n v - v\partial_n w)\,\mathrm{d}s(\sigma,\tau) = 0.$$

By the second Green identity (7.19) for the domain $\tilde{D}_\varepsilon$ which contains all points in D such that their distances from the poles (x,y) and (ξ,η) are larger than ε. We have

$$\int_{\partial B((x,y);\varepsilon)} (w\partial_n v - v\partial_n w)\mathrm{d}s(\sigma,\tau) = \int_{\partial B((\xi,\eta);\varepsilon)} (v\partial_n w - w\partial_n v)\mathrm{d}s(\sigma,\tau)\ . \tag{12.39}$$

Using the estimates (8.3)–(8.4) we infer

$$\lim_{\varepsilon\to 0}\int_{\partial B((x,y);\varepsilon)} |v\partial_n w|\mathrm{d}s(\sigma,\tau) = \lim_{\varepsilon\to 0}\int_{\partial B((\xi,\eta);\varepsilon)} |w\partial_n v|\mathrm{d}s(\sigma,\tau) = 0, \tag{12.40}$$

and

$$\lim_{\varepsilon\to 0}\int_{\partial B((x,y);\varepsilon)} w\partial_n v\,\mathrm{d}s(\sigma,\tau) = w(x,y), \quad \lim_{\varepsilon\to 0}\int_{\partial B((\xi,\eta);\varepsilon)} v\partial_n w\,\mathrm{d}s(\sigma,\tau) = v(\xi,\eta). \tag{12.41}$$

Letting $\varepsilon \to 0$ in (12.39) and using (12.40) and (12.41), we obtain

$$N(x,y;\xi,\eta) = w(x,y) = v(\xi,\eta) = N(\xi,\eta;x,y).$$

8.3 (a) The solution for the Poisson equation with zero Dirichlet boundary condition is given by

$$w(r,\theta) = \frac{\tilde{f}_0(r)}{2} + \sum_{n=1}^{\infty}[\tilde{f}_n(r)\cos n\theta + \tilde{g}_n(r)\sin n\theta]. \tag{12.42}$$

Substituting the coefficients $\tilde{f}_n(r)$, $\tilde{g}_n(r)$ into (12.42), we obtain

$$\begin{aligned} w(r,\theta) = {} & \frac{1}{2}\int_0^r K_1^{(0)}(r,a,\rho)\delta_0(\rho)\rho\,\mathrm{d}\rho + \frac{1}{2}\int_r^a K_2^{(0)}(r,a,\rho)\delta_0(r)\rho\,\mathrm{d}\rho \\ & + \sum_{n=1}^{\infty}\left(\int_0^r K_1^{(n)}(r,a,\rho)[\delta_n(\rho)\cos n\theta + \varepsilon_n(r)\sin n\theta]\rho\,\mathrm{d}\rho\right) \\ & + \sum_{n=1}^{\infty}\left(\int_r^a K_2^{(n)}(r,a,\rho)[\delta_n(r)\cos n\theta + \varepsilon_n(r)\sin n\theta]\rho\,\mathrm{d}\rho\right)\ . \end{aligned}$$

Recall that the coefficients $\delta_n(\rho)$, $\varepsilon_n(r)$ are the Fourier coefficients of the Function F, hence

$$\delta_n(\rho) = \frac{1}{\pi}\int_0^{2\pi} F(\rho,\varphi)\cos n\varphi\,\mathrm{d}\varphi, \quad \varepsilon_n(r) = \frac{1}{\pi}\int_0^{2\pi} F(\rho,\varphi)\sin n\varphi\,\mathrm{d}\varphi.$$

Substitute these coefficients, and interchange the order of summation and integration to obtain

$$w(r,\theta) = \int_0^a \int_0^{2\pi} G(r,\theta;\rho,\varphi)F(\rho,\varphi)\,\mathrm{d}\varphi\rho\,\mathrm{d}\rho,$$

where G is given by

$$G(r,\theta;\rho,\varphi) = \frac{1}{2\pi}\begin{cases} \log\frac{r}{a} + \sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{r}{a}\right)^n - \left(\frac{a}{r}\right)^n\right]\left(\frac{\rho}{a}\right)^n \cos n(\theta-\varphi) & \text{if } \rho < r, \\ \log\frac{\rho}{a} + \sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{\rho}{a}\right)^n - \left(\frac{a}{\rho}\right)^n\right]\left(\frac{r}{a}\right)^n \cos n(\theta-\varphi) & \text{if } \rho > r. \end{cases}$$

(b) To calculate the sum of the above series use the identities

$$\sum_{n=1}^{\infty}\frac{1}{n}z^n\cos n\alpha = \int_0^z \sum_{n=1}^{\infty}\zeta^{n-1}\cos n\alpha\,\mathrm{d}\zeta$$

$$= \int_0^z \frac{\cos\alpha - \zeta}{1+\zeta^2 - 2\zeta\cos\alpha}\,\mathrm{d}\zeta = -\frac{1}{2}\log(1+z^2-2z\cos\alpha).$$

8.5 (a) Let $(x,y),(\xi,\eta)\in\mathbb{R}^2_+$. The function $\Gamma(x-\xi,y+\eta)$ is harmonic as a function of (ξ,η) in $\mathbb{R}^2_+$, and therefore

$$\Delta_{(\xi,\eta)}G(x,y;\xi,\eta) = \Delta\Gamma(x-\xi,y-\eta) - \Delta\Gamma(x-\xi,y+\eta) = -\delta(x-\xi,y-\eta).$$

Since $G(x,y;\xi,0) = 0$, it follows that G satisfies all the desired properties of the Green function.

Notice that on the boundary of $\mathbb{R}^2_+$ the exterior normal derivative is $\partial/\partial y$. It is easy to verify that

$$\left.\frac{\partial G(x,y;\xi,\eta)}{\partial y}\right|_{y=0} = \frac{\eta}{\pi[(x-\xi)^2+\eta^2]} \qquad x\in\mathbb{R},\ (\xi,\eta)\in\mathbb{R}^2_+\,.$$

(b) Check that

$$G(x,y;\xi,\eta) = -\frac{1}{4\pi}\ln\left\{\frac{[(x-\xi)^2+(y-\eta)^2]\,[(x+\xi)^2+(y+\eta)^2]}{[(x-\xi)^2+(y+\eta)^2]\,[(x+\xi)^2+(y-\eta)^2]}\right\}$$

satisfies all the desired properties.

8.7 (a) Let u be a smooth function with a compact support in $\mathbb{R}^2$. We need to prove that

$$u_\varepsilon(\vec{y}) := \int_{\mathbb{R}^2}\rho_\varepsilon(\vec{x})u(\vec{x})\,\mathrm{d}\vec{x} \to u(\vec{y})$$

as $\varepsilon\to 0$, where

$$\rho_\varepsilon(\vec{x}) := \varepsilon^{-2}\rho\left(\frac{\vec{x}-\vec{y}}{\varepsilon}\right).$$

Recall that ρ_ε is supported in a ball of radius ε around $\vec{y}$ and satisfies

$$\int_{\mathbb{R}^2} \rho_\varepsilon(\vec{x})\,\mathrm{d}\vec{x} = 1.$$

By the continuity of u at y, it follows that for any $\delta > 0$ there exists $\varepsilon > 0$ such that $|u(\vec{x}) - u(\vec{y})| < \delta$ for all $\vec{x} \in B(y,\varepsilon)$. Therefore,

$$|u_\varepsilon(\vec{y}) - u(\vec{y})| = \left| \int_{\mathbb{R}^2} \rho_\varepsilon(\vec{x})\,[u(\vec{x}) - u(\vec{y})]\,\mathrm{d}\vec{x} \right|$$

$$\leq \int_{B(y,\varepsilon)} \rho_\varepsilon(\vec{x})|u(\vec{x}) - u(\vec{y})|\,\mathrm{d}\vec{x} < \delta \int_{B(y,\varepsilon)} \rho_\varepsilon(\vec{x})\,\mathrm{d}\vec{x} = \delta.$$

Thus, $\lim_{\varepsilon\to 0_+} u_\varepsilon(\vec{y}) = u(\vec{y})$.

(b) Since

$$2\pi \int_0^1 \exp\left(\frac{1}{|r|^2 - 1}\right) r\mathrm{d}A \approx 0.4665,$$

it follows that the normalization constant c for the function

$$\rho(\vec{x}) = \begin{cases} c\exp[1/(|\vec{x}|^2 - 1)] & |\vec{x}| \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

is approximately 2.1436.

The proof that ρ_ε is an approximation of the delta function (for this particular ρ) is the same as in part (a)

8.9 Fix $y \in \mathbb{R}$. Use Exercise 5.20 to show that as a function of (x,t) the kernel K solves the heat equation for $t > 0$.

Set

$$\rho(x) := \frac{1}{\sqrt{\pi}} \mathrm{e}^{-x^2}.$$

Then $\int_{-\infty}^{\infty} \rho(x)\,\mathrm{d}x = 1$. Consider

$$\rho_\varepsilon(x) := \varepsilon^{-1} \rho\left(\frac{x-y}{\varepsilon}\right).$$

By Exercise 8.7, ρ_ε approximates the delta function as $\varepsilon \to 0_+$.

Take $\varepsilon = \sqrt{4kt}$, where $t > 0$. Then $\rho_\varepsilon(x) = K(x,y,t)$. Therefore, for any smooth function $\phi(x)$ with a compact support in $\mathbb{R}$ we have

$$\lim_{t\to 0_+} \int_0^L K(x,y,t)\phi(x)\,\mathrm{d}x = \phi(y).$$

Thus, $K(x,y,0) = \delta(x-y)$.

8.11 Let $(x, y) \in D_R$, and let

$$(\tilde{x}, \tilde{y}) := \frac{R^2}{x^2 + y^2}(x, y)$$

be the reflection of (x, y) with respect to the circle ∂B_R. Set

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2},\ r^* = \sqrt{(x - \frac{R^2}{\rho^2}\xi)^2 + (y - \frac{R^2}{\rho^2}\eta)^2},\ \rho = \sqrt{\xi^2 + \eta^2}.$$

It is easy to verify (as was done in Exercise 8.1) that the function

$$G_R(x, y; \xi, \eta) = -\frac{1}{2\pi} \ln \frac{Rr}{\rho r^*} \qquad (\xi, \eta) \neq (x, y) \tag{12.43}$$

is the Green function in D_R.

8.13 Fix $(\xi, \eta) \in B_R$, and define for $(x, y) \in B_R \setminus (\xi, \eta)$

$$N_R(x, y; \xi, \eta) = \begin{cases} -\frac{1}{2\pi} \ln \frac{rr^*\rho}{R^3} & (\xi, \eta) \neq (0, 0), \\ -\frac{1}{2\pi} \ln \frac{r}{R} & (\xi, \eta) = (0, 0), \end{cases} \tag{12.44}$$

where

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2},\ r^* = \sqrt{(x - \frac{R^2}{\rho^2}\xi)^2 + (y - \frac{R^2}{\rho^2}\eta)^2},\ \rho = \sqrt{\xi^2 + \eta^2}.$$

It is easy to verify that

$$\Delta N_R(x, y; \xi, \eta) = -\delta(x - \xi, y - \eta),$$

and that N_R satisfies the boundary condition

$$\frac{\partial N_R(x, y; \xi, \eta)}{\partial r} = \frac{1}{2\pi R}\,.$$

Finally one has to check that N_R satisfies the normalization (8.34).

Chapter 9

9.1 (b) From the eikonal equation itself $u_z(0,0,0) = \pm\sqrt{1 - u_x^2(0,0,0) - u_y^2(0,0,0)} = \pm 1$, where the sign ambiguity means that there are two possible waves, one propagating into $z > 0$, and one into $z < 0$.

The characteristic curves (light rays) for the equation are straight lines (since the refraction index is constant) perpendicular to the wavefront (this is a general property of the characteristic curves). Therefore the ray that passes through $(0,0,0)$ is in the direction $(0,0,1)$. This implies $u_x(0,0,z) = u_y(0,0,z) = 0$ for all z, and hence $u_{xz}(0,0,z) = u_{yz}(0,0,z) = 0$. Differentiating the eikonal equation by z and using the last identity implies $u_{zz}(0,0,0) = 0$. The result for the higher derivatives is obtained similarly by further differentiation.

9.3 Verify that the proposed solution (9.26) indeed satisfies (9.23) and (9.25), and that $u_r(0,t) = 0$.

9.5 Use formula (9.26). The functions $u(r,0) = 2$ and $u_t(r,0) = 1 + r^2$ are both even which implies at once their even extension. Substitute the even extension into (9.26) and perform the integration to obtain $u(r,t) = 2 + (1 + r^2 + c^2t^2)t$.

9.7 The representation (9.35) for the spherical mean makes it easier to interchange the order of integration. For instance,

$$\frac{\partial}{\partial a} M_h(a, \vec{x}) = \frac{1}{4\pi} \int_{|\vec{\eta}|=1} \nabla h(\vec{x} + a\vec{\eta}) \cdot \vec{\eta} \, \mathrm{d}S_{\vec{\eta}}.$$

Using Gauss' theorem (recall that the radius vector is orthogonal to the sphere) we can express the last term as

$$\frac{a}{4\pi} \int_{|\eta|<1} \Delta_x h(\vec{x} + a\vec{\eta}) \, \mathrm{d}\vec{\eta}.$$

To return to a surface integral notation we rewrite the last expression as

$$\frac{a^{-2}}{4\pi} \Delta_x \int_{|\vec{x}-\vec{\xi}|<a} h(\vec{\xi}) \, \mathrm{d}\vec{\xi} = \frac{a^{-2}}{4\pi} \Delta_x \int_0^a \mathrm{d}\alpha \int_{|\vec{x}-\vec{\xi}|=\alpha} h(\vec{\xi}) \, \mathrm{d}S_{\vec{\xi}} =$$

$$a^{-2} \Delta_x \int_0^a \alpha^2 M_h(\alpha, \vec{x}) \, \mathrm{d}\alpha.$$

Multiplying the two sides by a^2 and differentiating again with respect to the variable a we obtain the Darboux equation.

9.9 Using the same method as in Subsection 9.5.2, one finds that

$$\lambda_{l,n,m} = \pi^2 \left(\frac{l^2}{a^2} + \frac{n^2}{b^2} + \frac{m^2}{c^2} \right), \quad u_{l,n,m}(x,y,z) = \sin \frac{l\pi x}{a} \sin \frac{n\pi y}{b} \sin \frac{m\pi z}{c},$$

for $l, n, m = 1, 2, \ldots$.

9.11 *Hint*: Differentiate (9.76) with respect to r to obtain one recursion formula, and differentiate with respect to θ to obtain another recursion formula. Combining the two recursion formulas leads to (9.77).

9.12 *Hint*: In part (a) you can use the recursion formula for Bessel functions. In part (b) use the integral representation for Bessel functions.

9.13 (a) The functions v_1 and v_2 satisfy the the Legendre equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(1-t^2)\frac{\mathrm{d}v_1}{\mathrm{d}t}\right] + \mu_1 v_1 = 0 \qquad -1 < t < 1, \tag{12.45}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(1-t^2)\frac{\mathrm{d}v_2}{\mathrm{d}t}\right] + \mu_2 v_2 = 0 \qquad -1 < t < 1. \tag{12.46}$$

Multiply (12.45) by v_2 and (12.46) by v_1, and subtract to obtain

$$v_2\frac{\mathrm{d}}{\mathrm{d}t}\left[(1-t^2)\frac{\mathrm{d}v_1}{\mathrm{d}t}\right] - v_1\frac{\mathrm{d}}{\mathrm{d}t}\left[(1-t^2)\frac{\mathrm{d}v_2}{\mathrm{d}t}\right] = (\mu_2-\mu_1)v_1v_2 \qquad -1 < t < 1. \tag{12.47}$$

Integrating (12.47) over $[-1, 1]$ implies

$$\int_{-1}^{1}\left\{v_2\left[(1-t^2)v_1'\right]' - v_1\left[(1-t^2)v_2'\right]'\right\}\mathrm{d}s = (\mu_2-\mu_1)\int_{-1}^{1} v_1(s)v_2(s)\,\mathrm{d}s.$$

Integrating the left hand side by parts taking into account that v_i are smooth and that $1-t^2$ vanishes at the end points, we obtain $(\mu_2-\mu_1)\int_{-1}^{1} v_1(s)v_2(s)\,\mathrm{d}s = 0$. Since $\mu_1 \neq \mu_1$ it follows that $\int_{-1}^{1} v_1(s)v_2(s)\,\mathrm{d}s = 0$.
(b) Suppose that Legendre equation admits a smooth solution v on $[-1, 1]$ with $\mu \neq k(k+1)$. By part (a), v is orthogonal to all Legendre polynomials, and by linearity v is orthogonal to the space of all polynomials. It follows from Weierstrass' approximation theorem that v is orthogonal to the space $E(-1, 1)$. This implies that $v = 0$.

9.15 Write the general homogeneous harmonic polynomial as in Corollary 9.24, and express it in the form $Q(r, \phi, \theta) = r^n F(\phi, \theta)$. Substitute Q into the spherical form of the Laplace equation (see 9.86), to get that F satisfies

$$\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\frac{\partial F}{\partial\phi}) + \frac{1}{\sin^2\phi}\frac{\partial^2 F}{\partial\theta^2} = -n(n+1)F.$$

Therefore F is a spherical harmonic (or combinations of spherical harmonics).

9.17 (a) By Exercise 9.13, Legendre polynomials with different indices are orthogonal to each other on $E(-1, 1)$. Furthermore, since P_n is an n-degree polynomial, we infer that $P_n(t)$ satisfies

$$\int_{-1}^{1} t^l P_n(t)\,\mathrm{d}t = 0 \qquad \forall l = 0, 1, 2, \ldots, n-1. \tag{12.48}$$

The characterization (12.48), together with the normalization $P_n(1) = 1$ determines the Legendre polynomials uniquely.

Set

$$Q_n(t) := \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left[\left(t^2 - 1\right)^n \right].$$

Clearly, Q_n is an n-degree polynomial. Repeatedly integrating by parts it follows that $\int_{-1}^{1} Q_n(s) Q_m(s)\,\mathrm{d}s = 0$ for $n \neq m$. Moreover, $Q_n(1) = 1$. Therefore, $P_n = Q_n$.

(b) We thus compute

$$\int_{-1}^{1} P_n(t)^2 \mathrm{d}t = \frac{1}{2^{2n} n!^2} \int_{-1}^{1} \left[\frac{\mathrm{d}^n}{\mathrm{d}t^n} (t^2 - 1)^n \right]^2 \mathrm{d}t = \frac{(2n!)}{2^{2n} n!^2} \int_{-1}^{1} (t^2 - 1)^n \,\mathrm{d}t = \frac{2}{2n+1}. \quad (12.49)$$

Returning to the general case of associated Legendre functions, and using (9.101) we write down

$$\int_0^{\pi} \left[P_n^m(\cos\phi)\right]^2 \sin\phi \,\mathrm{d}\phi = \int_{-1}^{1} \left[P_n^m(t)\right]^2 \,\mathrm{d}t = \int_{-1}^{1} \left[(1 - t^2)^{m/2} \frac{\mathrm{d}^m P_n}{\mathrm{d}t^m} \right]^2 \mathrm{d}t. \quad (12.50)$$

Performing m integrations by parts brings the integral into the form

$$(-1)^m \int_{-1}^{1} P_n \frac{\mathrm{d}^m}{\mathrm{d}t^m} \left[(1 - t^2)^m \frac{\mathrm{d}^m P_n}{\mathrm{d}t^m} \right] \mathrm{d}t.$$

Notice that the expression

$$Q(t) = \frac{\mathrm{d}^m}{\mathrm{d}t^m} \left[(1 - t^2)^m \frac{\mathrm{d}^m P_n}{\mathrm{d}t^m} \right]$$

is a polynomial of degree n. Moreover, the term t^n in this polynomial originates in the associated term $a_n t^n$ in the polynomial. A brief calculation shows that

$$Q(t) = (-1)^m \frac{(n+m)!}{(n-m)!} a_n t^n.$$

The orthogonality condition (12.48) implies that the only contribution to the integral comes through this term, namely,

$$\int_{-1}^{1} \left[P_n^m(t)\right]^2 \,\mathrm{d}t = \int_{-1}^{1} \frac{(n+m)!}{(n-m)!} a_n P_n(t) t^n \,\mathrm{d}t.$$

We use again (12.48) and (12.50) to finally obtain

$$\int_{-1}^{1} \left[P_n^m(t)\right]^2 \,\mathrm{d}t = \frac{2(n+m)!}{(2n+1)(n-m)!}.$$

9.19 Let B_R be the open ball with a radius R and a center at the origin. For $\vec{x} \in B_R$, denote by

$$\tilde{x} := \frac{R^2}{|\vec{x}|^2} \vec{x}$$

the inverse point of $\vec{x}$ with respect to the sphere ∂B_R. It is convenient to define the ideal point ∞ as the inverse of the origin.

Fix $\vec{y} \in B_R$. Recall that as a function of $\vec{x}$ the function $\Gamma(|\vec{x}-\vec{y}|)$ is harmonic for all $\vec{x} \neq \vec{y}$ and satisfies $-\Delta\Gamma(\vec{x};\vec{y}) = \delta(\vec{x}-\vec{y})$. Consequently,

$$\Gamma(\sqrt{(|\vec{x}||\vec{y}|/R)^2 + R^2 - 2\vec{x}\cdot\vec{y}}\,)$$

is harmonic in B_R. On the other hand, for $\vec{x} \in \partial B_R$ we have

$$G(\vec{x};\vec{y}) = \Gamma(\sqrt{|\vec{x}|^2 + |\vec{y}|^2 - 2\vec{x}\cdot\vec{y}}) - \Gamma(\sqrt{(|\vec{x}||\vec{y}|/R)^2 + R^2 - 2\vec{x}\cdot\vec{y}}\,) = 0.$$

Therefore, the Green function is given by

$$G(\vec{x};\vec{y}) = \Gamma(\sqrt{|\vec{x}|^2 + |\vec{y}|^2 - 2\vec{x}\cdot\vec{y}}) - \Gamma(\sqrt{(|\vec{x}||\vec{y}|/R)^2 + R^2 - 2\vec{x}\cdot\vec{y}}\,). \qquad (12.51)$$

Now, Let $\vec{y} \in \partial B_R$. Then

$$\frac{\partial G}{\partial n} = \frac{\partial G}{\partial |y|} = \frac{R^2 - |\vec{x}|^2}{N\omega_N R}|\vec{x}-\vec{y}|^{-N}.$$

9.21 (a) Suppose that u is harmonic in B_R, where B_R is the open ball of radius R centered at the origin in $\mathbb{R}^N$, and let $r < R$. Using (9.174) and (9.180), it follows that the Poisson integral formula for u is giving by

$$u(\vec{y}) = \frac{r^2 - |y|^2}{n\omega_N r}\int_{\partial B_r}\frac{u(\vec{x})}{|\vec{x}-\vec{y}|^N}\,\mathrm{d}\sigma_{\vec{x}}. \qquad (12.52)$$

Substituting $\vec{y} = \vec{0}$ in (12.52), we obtain

$$u(\vec{0}) = \frac{r^2}{n\omega_N r}\int_{\partial B_r}\frac{u(\vec{x})}{|\vec{x}|^N}\,\mathrm{d}\sigma_x = \frac{1}{n\omega_N r^{N-1}}\int_{\partial B_r} u(\vec{x})\,\mathrm{d}\sigma_x\,. \qquad (12.53)$$

(b) Let $0 < r < R$. We write $u(\vec{x}) = u(r\vec{\omega})$, where $r = |x|$ and $\vec{\omega} = \vec{x}/r$. We also define

$$U(r) := \frac{1}{n\omega_N r^{N-1}}\int_{\partial B_r} u(\vec{x})\,\mathrm{d}\sigma_{\vec{x}} = \frac{1}{n\omega_N}\int_{|\vec{\omega}|=1} u(r\vec{\omega})\,\mathrm{d}\vec{\omega}.$$

Differentiating with respect to r we obtain

$$U_r(r) = \frac{1}{n\omega_N}\int_{\partial B_r}\frac{\partial u(r\vec{\omega})}{\partial r}\,\mathrm{d}\vec{\omega} = \frac{1}{n\omega_N r^{N-1}}\int_{\partial B_r}\frac{\partial u(\vec{x})}{\partial r}\,\mathrm{d}\sigma_{\vec{x}} = 0\,.$$

Therefore,

$$U(r) = \text{constant} = \lim_{r\to 0} U(r) = u(\vec{0}).$$

(c) The proof of the strong maximum principle for domains in $\mathbb{R}^N$ is exactly the same as for planar domains, and therefore it is omitted.

The weak maximum principle is trivial for the constant function. Suppose now that D is bounded and u is a nonconstant harmonic function in D which is continuous on $\bar{D}$. Since $\bar{D}$ is compact, u achieves its maximum on $\bar{D}$. By the strong maximum principle, the maximum is achieved on ∂D.

9.23 (a) Write $\vec{x} = (x', x_N)$, and let $\tilde{x} := (x', -x_N)$ be the inverse point of $\vec{x}$ with respect to the hyperplane $\partial\mathbb{R}^N_+$. Fix $\vec{y} \in \mathbb{R}^N_+$. The function $\Gamma(\tilde{x}; \vec{y})$ is harmonic as a function of $\vec{x}$ in $\mathbb{R}^2_+$, while $\Delta_{\vec{x}}\Gamma(\vec{x}; \vec{y}) = -\delta(\vec{x} - \vec{y})$. Consider the function

$$G(\vec{x}; \vec{y}) := \Gamma(\vec{x}; \vec{y}) - \Gamma(\tilde{x}; \vec{y}).$$

Since for $\vec{x} \in \partial\mathbb{R}^N_+$ we have $G(\vec{x}; \vec{y}) = 0$, it follows that G is indeed the Green function on $\mathbb{R}^N_+$.

Notice that for $\vec{y} \in \partial\mathbb{R}^N_+$ the exterior normal derivative is $\partial/\partial y_N$. Hence,

$$\left.\frac{\partial G(\vec{x}; \vec{y})}{\partial \vec{n}_{\vec{y}}}\right|_{y_N=0} = \left.\frac{\partial G(\vec{x}; \vec{y})}{\partial y_N}\right|_{y_N=0} = \frac{2x_N}{N\omega_n|\vec{x} - \vec{y}|^N} \qquad \vec{x} \in \mathbb{R}^N_+,\ \vec{y} \in \partial\mathbb{R}^N_+.$$

9.25 (a) The eigenvalues and eigenfunctions of the problem are

$$\lambda_{n,m} = \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right), \quad u_{n,m}(x, y) = \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b},$$

for $n, m = 1, 2, \ldots$. Now use (9.178) to get the expansion.
(b) The eigenvalues and eigenfunctions of the problem are

$$\lambda_{n,m} = \left(\frac{\alpha_{n,m}}{a}\right)^2, \ u_{n,m} = J_n(\frac{\alpha_{n,m}}{a}r)(A_{n,m}\cos n\theta + B_{n,m}\sin n\theta) \quad n \geq 0, \quad m \geq 1.$$

Now use (9.178) to get the expansion.

Chapter 10

10.1 The first variation is $\delta K = 2\int_0^1 y'\psi' \mathrm{d}t$, where ψ is the variation function. Therefore the Euler-Lagrange equation is $y'' = 0$, and the solution is $y_M(t) = t$. Expanding fully the functional with respect to the variation ψ about $y = y_M$, we have $K(u_M + \psi) = K(u_M) + \int_0^1 (\psi')^2 \mathrm{d}t$. This shows that y_M is a minimizer, and it is indeed unique.

10.3 The Euler-Lagrange equation is $\Delta u - gu^3 = 0, \quad x \in D$, while u satisfies the natural boundary conditions $\partial_n u = 0$ on ∂D

10.5 (a) The action is

$$J = \int_{t_1}^{t_2} \int_D \left[\frac{1}{2} u_t^2 - \frac{1}{2} |\nabla u|^2 - V(u) \right] \mathrm{d}\vec{x}.$$

(b) Taking the first variation and equating it to zero we obtain the nonlinear Klein-Gordon equation $u_{tt} - \Delta u + V'(u) = 0$.

10.7 (a) Introducing a Lagrange multiplier λ, we solve the minimization problem

$$\min \left[\int_D |\nabla u|^2 \mathrm{d}x\mathrm{d}y + \lambda \left(1 - \int_D u^2 \mathrm{d}x\mathrm{d}y \right) \right],$$

for all u that vanish on ∂D. Equating the first variation to zero we obtain the Euler-Lagrange equation

$$\Delta u = -\lambda u \quad x \in D, \qquad u = 0 \quad x \in \partial D. \tag{12.54}$$

(b) To see the connection to the Rayleigh-Ritz formula (9.53), multiply (12.54) by u and integrate by parts. Use $\int_D u^2 \mathrm{d}x\mathrm{d}y = 1$, to get $\lambda = \int_D |\nabla u|^2 \mathrm{d}x\mathrm{d}y$. Therefore, the Lagrange multiplier λ is exactly the value of the functional $\int_D |\nabla u|^2 \,\mathrm{d}x\mathrm{d}y$ *at* the constrained minimizer. Consider now (9.53) and define a new function w associated with the minimizer v through $w = v/(\int_D v^2 \,\mathrm{d}x\mathrm{d}y)^{1/2}$. Substituting into (9.53) and observing that $\int_D w^2 \mathrm{d}x\mathrm{d}y = 1$, shows that the value of λ that we found in part (a) is equal to the first eigenvalue characterized by (9.53).

10.9 The eigenvalue problem is

$$X^{(iv)}(x) - \lambda X(x) = 0, \qquad X(0) = X'(0) = X(b) = X'(b) = 0.$$

Multiply both sides by X and integrate over $(0, b)$. Performing two integrations by parts and using the boundary conditions we derive

$$\int_0^b (X'')^2 \,\mathrm{d}x = \lambda \int_0^b X^2 \mathrm{d}x.$$

Therefore $\lambda > 0$.

The solution satisfying the boundary conditions at $x = 0$ is

$$X(x) = A\,(\cosh\alpha x - \cos\alpha x) + B\,(\sinh\alpha x - \sin\alpha x)\,.$$

Enforcing the boundary condition at $x = b$, we obtain that a necessary and sufficient condition for a nontrivial solution is indeed given by condition (10.73).

10.11 (a) Let $\{v_n\}$ be an orthonormal infinite sequence. Then $\|v_n\| = 1$, and therefore, $\{v_n\}$ is bounded.
(b) Let $v \in H$. By the Riemann-Lebesgue lemma (see (6.38)), we have

$$\lim_{n\to\infty}\langle v_n, v\rangle = 0 = \langle 0, v\rangle.$$

This shows that $\{v_n\}$ converges weakly to 0.
(c) Suppose that v is a strong limit of a subsequence of $\{v_{n_k}\}$. Then it is also the weak limit of this subsequence, and by part (c), $v = 0$. On the other hand, by the triangle inequality,

$$|\,\|v_{n_k}\| - \|v\|\,| \leq \|v_{n_k} - v\| \to 0.$$

But $\|v_n\| = 1$, therefore $\|v\| = 1$ and $v \neq 0$. Hence, $\{v_n\}$ does not admit any subsequence converging strongly to a function in H.

10.12 *Hint*: Suppose that $\{u_n\}$ weakly converges to u in $H_1(D)$. Then by Theorem 10.13, $\{u_n\}$ is a bounded sequence in $H_1(D)$. It follows that $\{u_n\}$ and $\{\partial u_n/\partial x_i\}$ are bounded sequences in $L_2(D)$, and therefore up to a subsequence, they converge weakly to $\tilde{u}$, and $\tilde{u}_i$ in $L_2(D)$, respectively. It remains to show that $\tilde{u}_i = \partial\tilde{u}/\partial x_i$.

Chapter 11

11.1 Expand u into a Taylor series at (x_i, y_j):

$$\begin{aligned}
u(x_{i+1}, y_{j+1}) &= u(x_i, y_j) + \partial_x u(x_i, y_j)\Delta x + \partial_y u(x_i, y_j)\Delta y \\
&+\frac{1}{2}\left[\partial_{xx} u(x_i, y_j)\Delta x^2 + 2\partial_{xy} u(x_i, y_j)\Delta x\Delta y + \partial_{yy} u(x_i, y_j)\Delta y^2\right] + \cdots ,\\
u(x_{i-1}, y_{j+1}) &= u(x_i, y_j) - \partial_x u(x_i, y_j)\Delta x + \partial_y u(x_i, y_j)\Delta y \\
&+\frac{1}{2}\left[\partial_{xx} u(x_i, y_j)\Delta x^2 - 2\partial_{xy} u(x_i, y_j)\Delta x\Delta y + \partial_{yy} u(x_i, y_j)\Delta y^2\right] + \cdots ,\\
u(x_{i-1}, y_{j-1}) &= u(x_i, y_j) - \partial_x u(x_i, y_j)\Delta x - \partial_y u(x_i, y_j)\Delta y \\
&+\frac{1}{2}\left[\partial_{xx} u(x_i, y_j)\Delta x^2 + 2\partial_{xy} u(x_i, y_j)\Delta x\Delta y + \partial_{yy} u(x_i, y_j)\Delta y^2\right] + \cdots ,\\
u(x_{i+1}, y_{j-1}) &= u(x_i, y_j) + \partial_x u(x_i, y_j)\Delta x - \partial_y u(x_i, y_j)\Delta y \\
&+\frac{1}{2}\left[\partial_{xx} u(x_i, y_j)\Delta x^2 - 2\partial_{xy} u(x_i, y_j)\Delta x\Delta y + \partial_{yy} u(x_i, y_j)\Delta y^2\right] + \cdots .
\end{aligned}$$

It follows at once that

$$U_{i+1,j+1} - U_{i-1,j+1} - U_{i+1,j-1} + U_{i-1,j-1} = 4\Delta x\Delta y\partial_{xy} u(x_i, y_j) + \cdots \quad ,$$

where

$$U_{i,j} = u(x_i, y_j).$$

Therefore, we obtain the following finite difference approximation for the mixed derivative:

$$\partial_{xy} u(x_i, y_j) = \frac{U_{i+1,j+1} - U_{i-1,j+1} - U_{i+1,j-1} + U_{i-1,j-1}}{4\Delta x\Delta y}.$$

11.3 To check the consistency of the Crank-Nicolson scheme we define for any function $v(x, t)$

$$R(v) = \frac{V_{i,n+1} - V_{i,n}}{\Delta t} - k\left(\frac{V_{i+1,n} - 2V_{i,n} + V_{i-1,n}}{2(\Delta x)^2} + \frac{V_{i+1,n+1} - 2V_{i,n+1} + V_{i-1,n+1}}{2(\Delta x)^2}\right),$$

where $V_{i,j} = v(x_i, y_j)$.

We now substitute the Taylor series expansion of the solution $u(x, t)$ into the heat equation in $R(u)$ and obtain

$$\begin{aligned}
R(u) &= \frac{1}{2}\Delta t\left[\partial_{tt} u(x_i, t_j) - k\partial_{xxt} u(x_i, t_j)\right] + \frac{1}{6}(\Delta t)^2\left[\partial_{ttt} u(x_i, t_j) - \frac{3}{2}k\partial_{xxtt} u(x_i, t_j)\right]\\
&-\frac{1}{12}(\Delta x)^2 k\partial_{xxxx} u(x_i, t_j) + \frac{1}{24}(\Delta t)^3 k\partial_{tttt} u(x_i, t_j).
\end{aligned}$$

It follows now that $\lim_{\Delta x,\Delta t\to 0} R(u) = 0$ and the scheme is indeed consistent.

11.5 The numerical solution: The finite difference equation for the Crank-Nicolson scheme is

$$\frac{U_{i,n+1}-U_{i,n}}{\Delta t}=\frac{U_{i+1,n}-2U_{i,n}+U_{i-1,n}}{2(\Delta x)^2}+\frac{U_{i+1,n+1}-2U_{i,n+1}+U_{i-1,n+1}}{2(\Delta x)^2}, \quad (12.55)$$

where $U_{i,n}=u(x_i,t_n)$, $1\le i\le N-2$, $n\ge 0$, and $N=(\pi/\Delta x)+1$. Notice that the boundary conditions determine the solution values at the endpoints, i.e.

$$U_{0,n}=U_{N-1,n}=0 \qquad n\ge 1.$$

The initial condition becomes

$$U_{i,0}=x_i(\pi-x_i) \qquad 0\le i\le N-1, \qquad x_i=i\Delta x.$$

Let us rewrite (12.55) as

$$U_{i,n+1}=\frac{\alpha}{2}(U_{i+1,n+1}-2U_{i,n+1}+U_{i-1,n+1})+r_{i,n}+U_{i,n},$$

where $\alpha=\Delta t/(\Delta x)^2$, $1\le i\le N-2$, $n\ge 0$, and

$$r_{i,n}=\frac{\alpha}{2}(U_{i+1,n}-2U_{i,n}+U_{i-1,n}).$$

We solve the algebraic equations with the Gauss-Seidel method.

The analytical solution: The general solution of the PDE is

$$u(x,t)=\sum_{n=1}^{\infty}B_n\,\mathrm{e}^{-n^2t}\,\sin(nx).$$

To find the coefficients B_n we expand $f(x)=x(\pi-x)$ into a sine series in $[0,\pi]$. We obtain

$$B_n=\frac{2}{\pi}\int_0^{\pi}x\,(\pi-x)\,\sin(nx)\,\mathrm{d}x=\begin{cases}0 & n=2m,\\ (8/\pi)(2m-1)^{-3} & n=2m-1.\end{cases} \quad (12.56)$$

Therefore, the analytical solution is

$$u(x,t)=\frac{8}{\pi}\sum_{m=1}^{\infty}\frac{\mathrm{e}^{-(2m-1)^2t}}{(2m-1)^3}\,\sin(2m-1)x.$$

We compare the analytical and numerical solutions at the point $(x,t)=(\pi/4,2)$. In the analytical solution we took partial sums with 2, 7 and 20 terms in the series, while in the numerical solution we used grids of size 25, 61 and 101. The time step is always $\Delta t=\Delta x/4$. The results are presented in the Table below. Notice that adding terms into the partial sums of the Fourier representation adds very little to the accuracy.

	Analytical solution			Numerical solution		
	2 first terms in the series	7 first terms in the series	20 first terms in the series	A mesh of 25 grid points	A mesh of 61 grid points	A mesh of 101 grid points
$u(\pi/4, 2)$	0.243689127	0.243689128	0.243689128	0.244344	0.243803	0.243756

11.6 (b) The Crank-Nicolson scheme for (11.74)–(11.74) is given by

$$\frac{U_{i,n+1} - U_{i,n}}{\Delta t} = \frac{U_{i+1,n} - 2U_{i,n} + U_{i-1,n}}{2(\Delta x)^2} + \frac{U_{i+1,n+1} - 2U_{i,n+1} + U_{i-1,n+1}}{2(\Delta x)^2},$$

where $U_{i,n} = u(x_i, t_n)$, $\quad 1 \le i \le N-1$, $\quad n \ge 0$, and $N = \pi/\Delta x + 1$. The initial condition leads to

$$U_{i,0} = f(x_i) \qquad 0 \le i \le N-2, \qquad x_i = i\Delta x.$$

Observe that the solution at the boundary point $x = 0$ is determined by the boundary condition

$$U_{0,n} = 0 \qquad n \ge 1.$$

We rewrite the equations in the form

$$U_{i,n+1} = \frac{\alpha}{2}(U_{i+1,n+1} - 2U_{i,n+1} + U_{i-1,n+1}) + r_{i,n} + U_{i,n},$$

where $\alpha = \Delta t/(\Delta x)^2$, $\quad 1 \le i \le N-2$, $\quad n \ge 0$ and

$$r_{i,n} = \frac{\alpha}{2}(U_{i+1,n} - 2U_{i,n} + U_{i-1,n}).$$

At the endpoint $x = 1$ we have a Neumann boundary condition. One option to eliminate from it an equation for $U_{N-1,n}$ is to approximate the derivative at $x = 1$ by a forward difference approximation. In this case we get

$$U_{N-1,n} = U_{N-2,n}.$$

Unfortunately this is a first order approximation and the error due to it might spoil the entire (second order) scheme. Therefore, it is beneficial to add an artificial point $U_{N,n}$, and to approximate the Neumann condition at $x = 1$ by $U_{N,n} = U_{N-2,n}$. Notice that now $U_{N-1,n}$ is an internal point.

11.7 The analytic solution: It is easy to see that $u(x,t) = t^5$ satisfies all the problem's conditions, and thus is the unique solution.
A numerical solution ($\Delta x = \Delta t = 0.1$):

$$\alpha = \frac{\Delta t}{(\Delta x)^2} = \frac{0.1}{0.1^2} = 10, \quad N = \frac{1}{\Delta x} + 1 = 11, \quad t_n = \frac{n}{10}, \quad x_i = \frac{i}{10}.$$

Let us write an explicit finite difference scheme:

$$\begin{cases} U_{i,0} &= 0 & 0 \leq i \leq 10, \\ U_{i,n+1} &= U_{i,n} + 10(U_{i+1,n} - 2U_{i,n} + U_{i-1,n}) + 5t_{n+1}^4 & 1 \leq i \leq 9, \quad n \geq 0, \\ U_{0,n+1} &= t_{n+1}^5 & n \geq 0, \\ U_{10,n+1} &= t_{n+1}^5 & n \geq 0. \end{cases} \tag{12.57}$$

The analytic solution takes the value $u(1/2, 3) = 243$ at the required point. Simulating the scheme (12.57) provides the value $u(1/2, 3) = 2.4 \cdot 10^{39}$. The numerical solution is not convergent since the scheme is unstable when $\Delta t \leq 0.5/(\Delta x)^2$.

11.9 Let (i, j) be the index of an internal maximum point. Both terms in the left hand side of (11.27) are dominated by $U_{i,j}$. Therefore, if $U_{i,j}$ is positive, the left hand side is negative which is a contradiction.

11.13 Let $p_i,\ i = 1, ..., 4(N-2)$ be the set of boundary point. For each i define the harmonic function T_i, such that $T_i(p_i) = 1$, while $T_i(p_j) = 0$ if $j \neq i$. Clearly the set $\{T_i\}$ spans all solutions to the Laplace equation in the grid. It also follows directly from the construction that the set $\{T_i\}$ is linearly independent.